# Circle Relationships
## An ADE Mathematics Lesson
## Days 31-40

| | |
|---|---|
| **Author** | ADE Content Specialists |
| **Grade Level** | 10th grade |
| **Duration** | Ten days |

**Aligns To**

**Mathematics HS:**
**Strand 1: Number and Operations**
**Concept 2: Numerical Operations**
**PO 3.** Calculate powers and roots of rational and irrational numbers.

**Strand 3: Patterns, Algebra, and Functions**
**Concept 3: Algebraic Representations**
**PO 1.** Create and explain the need for equivalent forms of an equation or expression.
**PO 2.** Solve formulas for specified variables.

**Strand 4: Geometry and Measurement**
**Concept 1: Geometric Shapes**
**PO 1.** Use the basic properties of a circle (relationships between angles, radii, intercepted arcs, chords, tangents, and secants) to prove basic theorems and solve problems.
**PO 6.** Solve problems using angle and side length relationships and attributes of polygons.
**Concept 4: Measurement**
**PO 2.** Find the length of a circular arc; find the area of a sector of a circle.

**Strand 5: Structure and Logic**
**Concept 1: Algorithms and Algorithmic Thinking**
**PO 1.** Select an algorithm that explains a particular mathematical process; determine the purpose of a simple mathematical algorithm.
**Concept 2: Logic, Reasoning, Problem Solving, and Proof**
**PO 1.** Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.
**PO 5.** Summarize and communicate mathematical ideas using formal and informal reasoning.
**PO 7.** Find structural similarities within different algebraic expressions and geometric figures.

**Connects To**

**Mathematics HS:**
**Strand 1: Number and Operations**
**Concept 3: Estimation**
**PO 1.** Determine rational approximations of irrational numbers.

**Strand 4: Geometry and Measurement**
**Concept 4: Measurement**
**PO 1.** Use dimensional analysis to keep track of units of measure when converting.

**Strand 5: Structure and Logic**
**Concept 2: Logic, Reasoning, Problem Solving, and Proof**
**PO 3.** Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.

| Aligns To | Connects To |
| --- | --- |
| **Strand 5: Structure and Logic**<br>**Concept 2: Logic, Reasoning, Problem Solving, and Proof**<br>**PO 12.** Construct a simple formal deductive proof.<br>**PO 13.** Identify and explain the roles played by definitions, postulates, propositions and theorems in the logical structure of mathematics, including Euclidean geometry. | |

## Overview

In this lesson, you will study circles and their relationships to angles, chords, secants, and other geometric properties.

## Purpose

Circles form parts of many other relationships and are the basis for many mathematical algorithms such as the unit circle found in trigonometry. It is important to have a good knowledge of the basic properties of a circle and their relationship to chords, tangents, angles, secants, radii, and similar geometric ideas.

## Materials

- Circle relationship worksheets
- Ruler/Protractor
- Compass

## Objectives

**Students will:**

- Use the basic properties of a circle (relationships between angles, radii, intercepted arcs, chords, tangents, and secants) to prove basic theorems and solve problems.
- Solve problems using angle and side length relationships and attributes of polygons.
- Find the length of a circular arc; find the area of a sector of a circle.

## Lesson Components

**Prerequisite Skills:** In earlier grades, you learned the fundamentals of circles. You learned how to find the radius, diameter, circumference, and area of a circle. You learned that concentric circles are circles that contain the same center but have different radii. You learned that vertical angles are congruent. You learned how to construct a circle using a compass.

**Vocabulary:** *circle, central angle, inscribed angle, vertical angle, chord, secant, tangent, radius, intercept arc, sector of a circle*

**Session 1 (5 days)**

1. Review basic circle relationships and find the measures of central angles, chords, and inscribed angles. Find the area of a sector of a circle.

**Session 2 (5 days)**

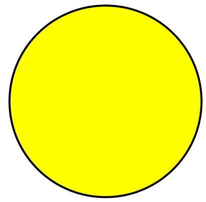
1. Find the measures of an intersecting tangent and its chord, angles formed by two secants or intersecting outside a circle, a tangent and a secant intersecting outside a circle, and two tangents intersecting outside a circle.

## Assessment

There are two assessments that will help pinpoint misconceptions before moving on to more complex comparisons. The first assessment comes after Session 1 and addresses basic circle relationships. The second assessment comes after Session 2 and assesses knowledge of lines that intersect either on the circumference of a circle or in a circle.

## Circle Relationships
## Session 1

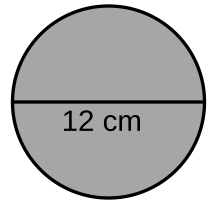
**Consider the following circle:**



**Define each of the following terms in relationship to the circle in your own words.**

| Term | Definition |
|---|---|
| Radius | |
| Diameter | |
| Circumference | |
| Center of the circle | |
| Circumference of circle | |
| Area of circle | |

**Example 1:** Let's review how to find the circumference and area of a circle. Find the circumference and area of a circle whose diameter is 12 cm.




**Solution:** If the diameter of the circle is $12\,cm$, the radius of the circle is $\frac{12}{2} = 6\ cm$.

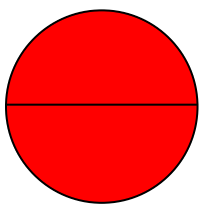
The circumference of the circle is given by either $C = 2\pi r$ or $C = \pi d$.

$$C = \pi d \Rightarrow C = \pi \bullet 12\ cm \Rightarrow C = 12\pi\ \ cm.$$

The area of the circle is given by $A = \pi r^2$.

$$A = \pi \bullet 6^2 = 36\pi\ cm^2$$

**Example 2:**



A **degree** is a unit of measure based on dividing a circle into 360 equal parts. Degrees are used to measure angles, arcs and rotations.

How many degrees are there in a circle? _____________

Each half of a circle is called a **semicircle.** How many degrees are there in a semicircle?

______________

What if we wanted to find the area of the semicircle that had a diameter of 12? Can you think of a way we could find the area?

A **central angle** is an angle whose vertex is the center of a circle and whose sides (rays) are radii.

A **sector of a circle** is a region bounded by a central angle and its arc.

We can find the area of a sector of a circle by setting up a proportion.

$$\frac{\text{central angle}}{360°} = \frac{\text{area}_{\text{sector}}}{\text{area}_{\text{circle}}}$$

**To find the area of the semicircle, set up the proportion.**

$$\frac{\text{central angle}}{360°} = \frac{\text{area}_{\text{sector}}}{\text{area}_{\text{circle}}} \Rightarrow \frac{180°}{360°} = \frac{\text{area}_{\text{sector}}}{36\pi\ \text{cm}^2} \Rightarrow \frac{1}{2} = \frac{\text{area}_{\text{sector}}}{36\pi\ \text{cm}^2}$$

$$\mathbf{\frac{1}{2} = \frac{area_{sector}}{36\pi\ cm^2} \Rightarrow 2 \bullet area_{sector} = 36\pi\ cm^2 \Rightarrow area_{sector} = \frac{36\pi\ cm^2}{2}}$$

$$\text{area}_{\text{sector}} = 18\pi\ \text{cm}^2$$

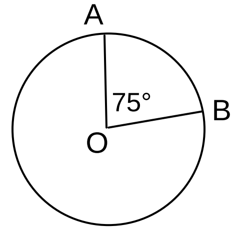
Summarize how to find the area of a semi-circle.

|   |
|---|
|   |

The measure of a central angle is the measure of its intercepted arc. An **intercepted arc** is that part of a circle that lies between two segments, rays, or secants that intersect the circle.

**Example 3:**

Find the measure of a central angle $\angle AOB$ whose intercepted arc $\overset{\frown}{AB}$ is 75°.



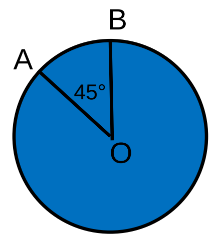

**Solution:**

| |
|---|
| The measure of a central angle equals the measure of its intercepted arc. |

Therefore, the measure of $\angle AOB$ is 75°.

**Example 4:**

Find the measure of an intercepted arc, $\overset{\frown}{AB}$ whose central angle $\angle AOB$ is 45°.




**Solution:**

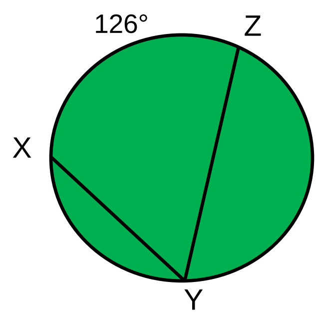
The central angle equals the measure of its intercepted arc. Therefore, the measure of $\overset{\frown}{AB}$ is 45°.

**Example 5:**

A **chord** is a segment whose endpoints are on a given circle. An **inscribed angle** is an angle with its vertex on the circle and with sides (rays) that are chords of the circle.

| The measure of an inscribed angle is one half of its intercepted arc. |
|---|

Find the measure of inscribed angle, $\angle XYZ$.




**Solution:**

The measure of the inscribed angle $\angle XYZ$ $(m\angle XYZ)$ equals one half the measure of its intercepted arc, $\overset{\frown}{XZ}$.

$$\frac{1}{2} \bullet 126° = 63°. \text{ Therefore, } m\angle XYZ = 63°.$$

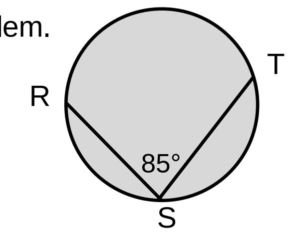
**Example 6:**

The measure of an inscribed angle of a circle is $85°$.

- What is the measure of its intercepted arc?
- What is the measure of any central angle with the same intercepted arc?

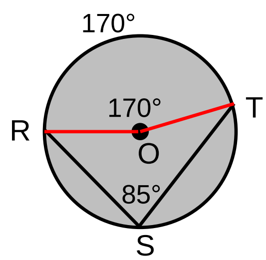
**Solution:**

First draw a picture to illustrate the problem.




Since the measure of the inscribed angle is one half of its intercepted arc, the measure of an intercepted arc would be twice its inscribed angle.

The measure of $\overset{\frown}{RT}$ is 85° • 2 = 170°.



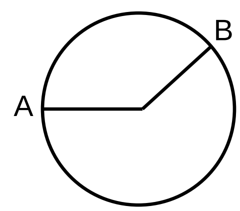

The measure of a central angle is equal to the measure of its intercepted arc. Therefore, the measure of any central angle with an intercepted arc of $170°$ would be $170°$.

$$m\angle ROT = 170°$$

**Now it's your turn to try some problems. Show your work in the space provided and draw diagrams when necessary to help you solve a problem.**

1. Find the area and circumference of a circle whose diameter is 8 cm.

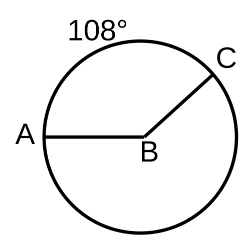
2. How many degrees are there in the angle shown in the diagram if the intercepted arc $\overset{\frown}{AB}$ is $\frac{1}{3}$ of the circumference of the circle?



3. Find the measure of a central angle whose intercepted arc is 52°.

4. Find the measure of an intercepted arc, $\overset{\frown}{AB}$ whose central angle $\angle AOB$ is 71°.

5. Find the measure of inscribed angle $\angle ABC$ .



6. The measure of an inscribed angle of a circle is 46°.

- What is the measure of its intercepted arc?
- What is the measure of any central angle with the same intercepted arc?

# Circle Relationships
# Assessment 1

**Problem 1: Define the following terms in your own words.**

| Term | Definition |
|---|---|
| Chord | |
| Central angle | |
| Inscribed angle | |
| Intercepted arc | |
| Degree | |
| Semicircle | |
| Sector of a circle | |

**Problem 2:**

Find the area of a sector of a circle whose central angle measures $45°$ and whose radius is 7 cm. Draw a diagram and show your work in the space provided.

**Problem 3:**

An inscribed angle has a measure of $23°$. Find both its intercepted arc and any central angle with that intercepted arc. Draw a diagram and show your work in the space provided.

## Circle Relationships
## Session 2

| Circle Definitions | |
|---|---|
| Major Arc of a circle | an arc whose measure is greater than 180 degrees |
| Minor Arc of a circle | an arc whose measure is less than 180 degrees |
| Chord of a circle | a segment whose endpoints are on a given circle |
| Secant of a circle | a line that intersects a circle at two points |
| Sector of a circle | a region bounded by a central angle and its arc |
| Tangent of a circle | a line in the plane of a circle that intersects the circle at exactly one point |

### Circle Formulas

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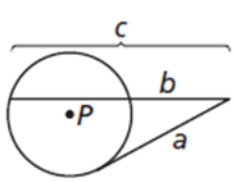
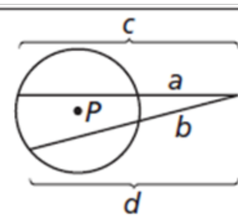
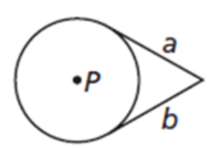
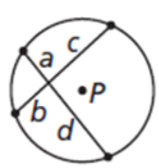
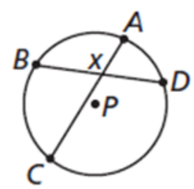
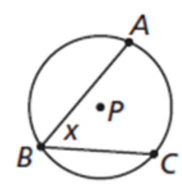
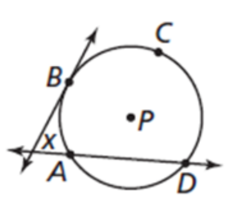
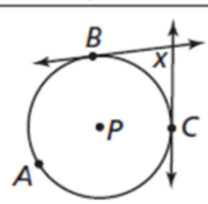
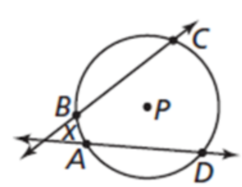
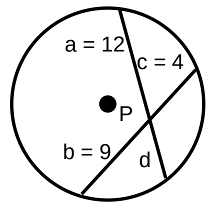
|---|---|---|
|  $\frac{c}{a} = \frac{a}{b}$ or $a^2 = bc$ |  $\frac{d}{a} = \frac{c}{b}$ or $ac = bd$ |  $a = b$ |
|  $\frac{a}{b} = \frac{c}{d}$ or $ad = bc$ |  $m\angle x = \frac{1}{2}\left(m\overset{\frown}{AB} + m\overset{\frown}{CD}\right)$ |  $m\angle x = \frac{1}{2}m\overset{\frown}{AC}$ |
|  $m\angle x = \frac{1}{2}\left(m\overset{\frown}{BCD} - m\overset{\frown}{AB}\right)$ |  $m\angle x = \frac{1}{2}\left(m\overset{\frown}{BAC} - m\overset{\frown}{BC}\right)$ |  $m\angle x = \frac{1}{2}\left(m\overset{\frown}{CD} - m\overset{\frown}{AB}\right)$ |

In many instances, we need to find the length of a chord, tangent, or secant. There are several different formulas to do this. Some of the circle formulas are just special cases of a more general form.

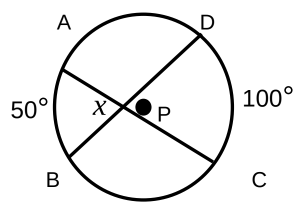
**Example 1:** Let's examine two chords that intersect in the interior of the circle. Find the measure of the missing chord in the diagram.




**Solution:** We know that we can set up a proportion according to the formula $\frac{a}{b}=\frac{c}{d}$ or that $ad = bc$. Therefore, it follows that $ad = bc \Rightarrow 12 \bullet d = 9 \bullet 4 \Rightarrow 12d = 36 \Rightarrow d = 3$.

We could have also set up the proportion $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{12}{9}=\frac{4}{d} \Rightarrow 12d = 36 \Rightarrow d = 3$.

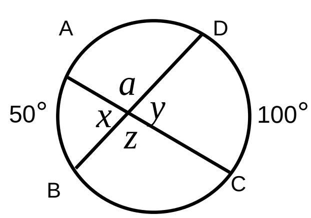
**Example 2:** Find the measure of $\angle x$ in circle P below.




**Solution:** $m\angle x = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$

$$m\angle x = \frac{1}{2}(50° + 100°) = \frac{1}{2}(150°) = 75°$$

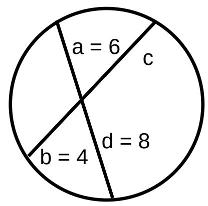
Note that we now know the measures of all the angles shown in this figure.



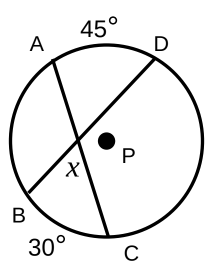

Since vertical angles are congruent, it follows that $m\angle x = m\angle y = 75°$ and $m\angle z = m\angle a = 105°$.

**Now you try some problems:**

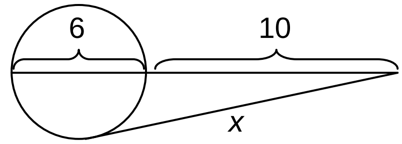
**Problem 1:** Find the measure of the missing chord in the diagram. Show your work in the space provided.



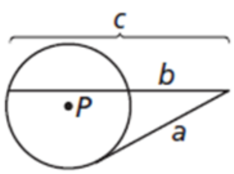

**Problem 2:** Find the measure of $\angle x$ in circle P below.



**Example 3:** Find $x$ in the picture below.



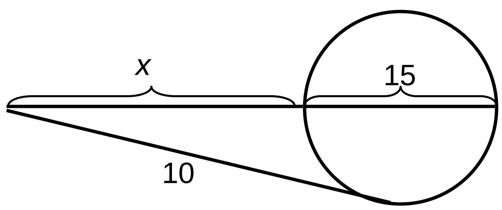

**Solution:** The diagram and formula below shows the relationship between the lengths of the line segments.




$\frac{c}{a} = \frac{a}{b}$ or $a^2 = bc$

$$\frac{10+6}{x} = \frac{x}{10} \Rightarrow (10+6)\bullet 10 = x \bullet x \Rightarrow 16 \bullet 10 = x^2 \Rightarrow x^2 = 160 \Rightarrow x = 4\sqrt{10}$$

**Example 4:**




**Solution:**

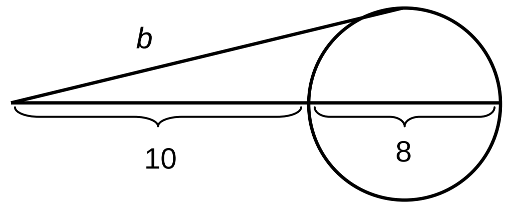
$$\frac{x+15}{10} = \frac{10}{x} \Rightarrow (x+15)\bullet x = 10 \bullet 10 \Rightarrow x^2 + 15x = 100 \Rightarrow x^2 + 15x - 100 = 0$$

$$x^2 + 15x - 100 = 0 \Rightarrow (x+20)(x-5) = 0$$

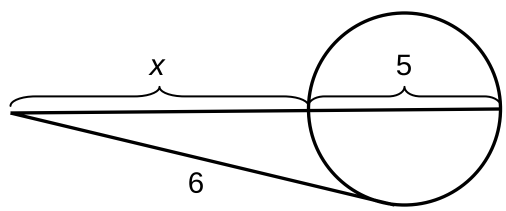
$$(x+20)(x-5) = 0 \Rightarrow x + 20 = 0 \text{ and } x - 5 = 0 \Rightarrow x = -20,\ 5$$

Since $x$ is a distance it cannot be negative so the correct answer is $x = 5$.

**Problem 3:** Find the value of $b$ in the diagram below. Show your work in the space provided.

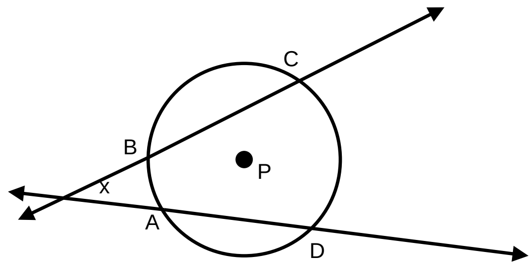



**Problem 4:** Find the measure of $x$ in the diagram below. Show all your work in the space provided.



**Example 5:**

Find the measure of $\angle x$ in the diagram below. $m\overset{\frown}{CD} = 100°$ and $m\overset{\frown}{AB} = 25°$

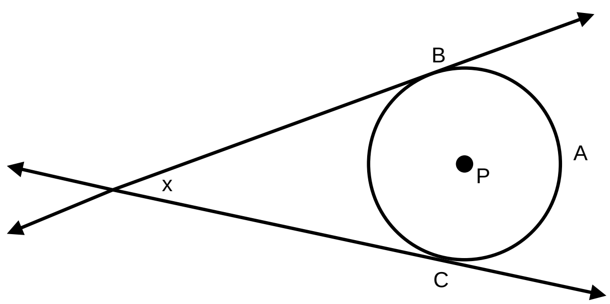



**Solution:**

$$m\angle x = \frac{1}{2}(m\overset{\frown}{CD} - m\overset{\frown}{AB}) \Rightarrow m\angle x = \frac{1}{2}(100° - 25°) \Rightarrow m\angle x = \frac{1}{2}(75°) = 37.5°$$

**Example 6:**

Find the measure of $\angle x$ in the diagram below. $m\overset{\frown}{BAC} = 236°$ and $m\overset{\frown}{BC} = 124°$

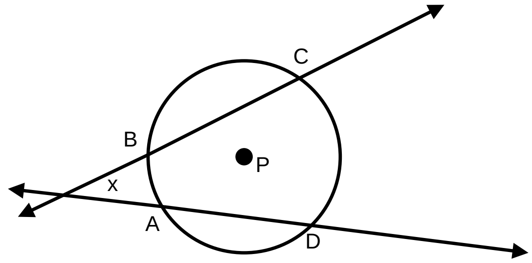



**Solution:**

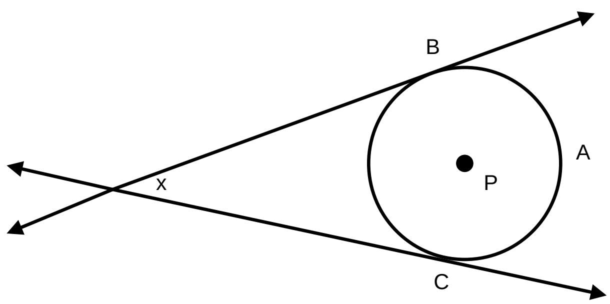
$$m\angle x = \frac{1}{2}m\overset{\frown}{BAC} - m\overset{\frown}{BC} \Rightarrow m\angle x = \frac{1}{2}(236° - 124°) \Rightarrow m\angle x = \frac{1}{2}(112°) = 56°$$

**Problem 5:** Find the measure of $\angle x$ in the diagram below. $m\overset{\frown}{CD} = 125°$ and $m\overset{\frown}{AB} = 37°$

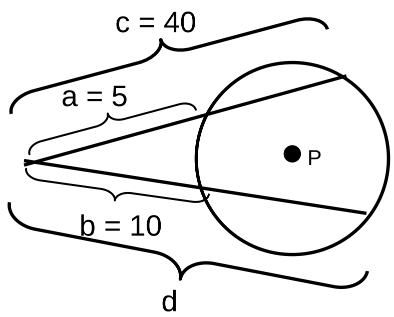
Show your work in the space provided.




**Problem 6:** Find the measure of $\angle x$ in the diagram below. $m\overset{\frown}{BAC} = 222°$ and $m\overset{\frown}{BC} = 138°$



**Example 7:** Find the measure of *d* in the diagram below if a = 5, b = 10, and c = 40.




**Solution:** $\frac{d}{a} = \frac{c}{b} \Rightarrow ac = bd \Rightarrow 5 \bullet 40 = 10 \bullet d \Rightarrow 200 = 10d \Rightarrow d = 20$

We have used formulas that involve chords, secants and tangents. Define each of these terms in your own words in the space provided.

| Term | Definition |
|---|---|
| Chord | |
| Tangent | |
| Secant | |

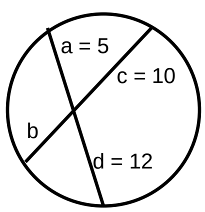
# Circle Relationships
## Assessment 2

1. Complete the following chart justifying your answers in the space provided.

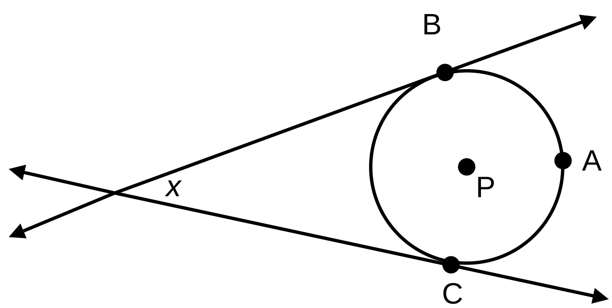
| |
|---|
| The difference between a chord and a secant is |
| The difference between a central angle and an inscribed angle is |
| The difference between a tangent and a secant is |
| The difference between a major arc and a minor arc is |

Find the measures of the missing chords, tangents, or secants in the following problems showing all your work in the space provided.

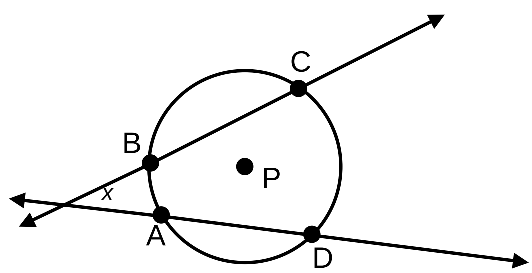
1. Find the measure of the missing chord in the diagram.



2. Find the measure of $\angle x$ in the diagram below. $\overparen{BAC} = 205°$ and $\overparen{BC} = 155°$.




3. Find the measure of $\angle x$ in the diagram below. $\overparen{CD} = 126°$ and $\overparen{AB} = 33°$.



## Extensions

1. Find central angles, inscribed angles, intercepted arcs, etc. when the measurements are given as algebraic expressions.

2. Explore relationships between central angles and intercepted arcs using an interactive applet.

   http://www.analyzemath.com/Geometry/CentralInscribedAngle/CentralInscribedAngle.html

## Sources


1999 Bringing the NCTM Standards to Life, Exemplary Practices from High Schools, p. 9-16
2000 NCTM Principles and Standards, p. 308-318
2008 AZ Department of Education Mathematics Standards
2008 The Final Report of the National Mathematics Advisory Panel, p. 29-30