APPENDIX 6-G. DERIVATION OF TCC AND TIF EQUATIONS FOR NCSC SPECIAL ITEM TYPES

In addition to dichotomously scored individual items, the NCSC ELA tests had special groups of items that were scored at the group level even though the items were all independently administered. The items groups that required these special scoring rules included foundational items sets and Tier 1 writing prompt items sets. For these item sets, the dichotomous two-parameter logistic (2PL) IRT model was fitted to the separate independent items. In order to obtain the test characteristic curve (TCC) and test information function (TIF) equations for each of these item sets, the special scoring rules for each type of item set had to be taken into account. The purpose of this appendix is to describe the derivation of the TCC and TIF equations for each type of item set in regard to their special scoring rules.

Foundational Item Sets

Foundational item sets appear in the grades 3 and 4 ELA assessments. A foundational item set consists of either three or five dichotomously scored (i.e., "correct" vs. "incorrect") SR items. For Pilot 2, the rules for assigning final scores for a foundational item set were (hereinafter referred to as the "combined" approach):

For Tier 1 foundational item sets, a student had to correctly respond to 3 out of 3 items to obtain a score of 1; otherwise, a score of 0 was assigned.

For Tiers 2 through 4 foundational item sets, a student had to correctly respond to 4 or 5 out of 5 items to obtain a score of 1; otherwise, a score of 0 was assigned.

TCC and TIF for Tier 1

Let $P_1(\theta)$, $P_2(\theta)$, and $P_3(\theta)$ represent the probability of a correct response, conditional on the IRT ability parameter θ, for each of the three items in a Tier 1 foundational item set. Let X represent a student's score on a Tier 1 foundational item set. By definition, the TCC is given by:

$$TCC = E(X|\theta).$$

Because the foundational item sets are dichotomously scored, X will be either 0 or 1. Thus,

$$TCC = E(X|\theta) = P(X=1|\theta).$$

For Tier 1, a score of 1 corresponds to giving the correct response to all three items. Because the items are independent, conditional on θ, the probability of giving the correct response to all three items is equal to the product of the probabilities of giving a correct response to each individual item, which results in:

$$TCC = E(X|\theta) = P(X=1|\theta) = P_1(\theta)\, P_2(\theta)\, P_3(\theta) \qquad (1)$$

Next we turn to the derivation of the equation for TIF for the Tier 1 foundational items. Following the definition given in Birnbaum (1968), the information function for score $X(\theta)$ is given by,

$$I(\theta) = \{\frac{d}{d\theta}E[X(\theta)]\}^2 / Var[X(\theta)]. \qquad (2)$$

First, consider the term in the numerator, the square of the derivative of $E[X(\theta)]$ with respect to θ. Because $E[X(\theta)]$ is the TCC, calculating the derivative of $E[X(\theta)]$ with respect to θ is the same as taking the

derivative of the TCC with respect to θ. Because the TCC is simply the product of probabilities, each of which depends on θ, taking the derivative of the TCC is easily accomplished by the chain rule.

Next consider the term in the denominator, Var[X(θ)]. Because X is dichotomous, its variance is given by P(X=1|θ) [1 – P(X=1|θ)]. As noted above, P(X=1|θ) is merely the equation for the TCC. Thus, following the steps outlined in this paragraph, the final equation for the TIF can be readily obtained.

Letting $Q_i(\theta)$ represent the probability of an incorrect response on the ith item (i.e., $Q_i(\theta) = 1 - P_i(\theta)$), the equation for I(θ) is thus given as follows,

$$I(\theta) = \frac{[DP_1(\theta)P_2(\theta)P_3(\theta)\{a_1Q_1(\theta)+a_2Q_2(\theta)+a_3Q_3(\theta)\}]^2}{P_1(\theta)P_2(\theta)P_3(\theta)\{1-P_1(\theta)P_2(\theta)P_3(\theta)\}}, \quad (3)$$

where D = 1.7 a scaling constant in the 2PL formula and a_i is the discrimination parameter for item i in the 2PL formula.

TCC and TIF for Tiers 2, 3, and 4

Let $P_1(\theta)$, $P_2(\theta)$, $P_3(\theta)$, $P_4(\theta)$ and $P_5(\theta)$ represent the conditional probability of a correct response for each of the five items in a foundational item set for Tiers 2, 3, and 4. Again, let X represent a student's score on a foundational item set. Because X is again dichotomously scored, the basic TCC equation remains the same as for Tier 1:

TCC = E(X|θ) = P(X=1|θ).

For Tiers 2, 3, and 4, a score of 1 is earned by either giving the correct response to all five items or by giving the correct response to any four out of five items. Because the items are independent, conditional on θ, the probability of giving the correct response to all five items is equal to the product of the probabilities of giving a correct response to each individual item. Similarly, the probability of correctly responding to four out of five items is equal to the product of five probabilities, four corresponding to correct responses and one corresponding to an incorrect response. Because there are five ways to correctly answer four out of five items, the total number of terms in the TCC equation will be six: one term corresponding to the case of getting all five correct and five terms corresponding to the five ways of getting four out of five correct. Letting $Q_i(\theta)$ represent the probability of an incorrect response on the ith item (i.e., $Q_i(\theta) = 1 - P_i(\theta)$), the equation for the TCC is given as follows:

$$\begin{aligned} TCC = E(X|\theta) = P(X=1|\theta) = {} & P_1(\theta)\,P_2(\theta)\,P_3(\theta)\,P_4(\theta)\,P_5(\theta) + Q_1(\theta)\,P_2(\theta)\,P_3(\theta)\,P_4(\theta)\,P_5(\theta) + \\ & P_1(\theta)\,Q_2(\theta)\,P_3(\theta)\,P_4(\theta)\,P_5(\theta) + P_1(\theta)\,P_2(\theta)\,Q_3(\theta)\,P_4(\theta)\,P_5(\theta) + \\ & P_1(\theta)\,P_2(\theta)\,P_3(\theta)\,Q_4(\theta)\,P_5(\theta) + P_1(\theta)\,P_2(\theta)\,P_3(\theta)\,P_4(\theta)\,Q_5(\theta) \end{aligned} \quad (4)$$

Next we turn to the derivation of the TIF equation. The basic definition is the same as for Tier 1:

$I(\theta) = \{\frac{d}{d\theta}E[X(\theta)]\}^2 / Var[X(\theta)]$.

As noted for Tier 1, E[X(θ)] is the TCC; thus, we must calculate the derivative of the TCC with respect to θ. In this case, the TCC is the sum of six terms, but each term is simply the product of conditional

probabilities. Thus, as with Tier 1, taking the derivative of the TCC is easily accomplished by applying the chain rule to each term. Next we again consider the term in the denominator, $Var[X(\theta)]$. As with Tier 1, because X is dichotomous, its variance is again given by $P(X=1|\theta)\ [1 - P(X=1|\theta)]$, where $P(X=1|\theta)$ is given by the TCC. Because the TCC equation is much longer for Tiers 2, 3, and 4 than for Tier 1, the final detailed TIF equation is also much longer and is omitted here to save space.

Tier 1 Writing Prompt Item Sets

Tier 1 writing prompt (WP) item sets were a part of ELA tests for all grades. Four to six independently correct/incorrect scored selected response (SR) items were grouped together to form each Tier 1 WP item set, where the number of SR items in each set varied depending on the grade level. A score of "0" was assigned if a student got no items correct in an item set, a score of "1" was assigned if a student got 1 or 2 items correct in an item set, and a score of "2" was assigned if student got 3 or more items correct in an item set. The scoring rule was applied to all Tier 1 WP item sets regardless of the number of items in the item set.

TCC and TIF for four-item sets

Let $P_1(\theta)$, $P_2(\theta)$, $P_3(\theta)$ and $P_4(\theta)$ represent the probability of a correct response, conditional on θ, for each of the four items in a Tier 1 WP item set. Let X represent a student's score on a Tier 1 WP item set. By definition, the TCC is given by:

$$TCC = E(X|\theta).$$

For the four-item sets, X=0 if a student correctly responds to 0 items, X=1 if a student correctly responds to either 1 or 2 items, and X=2 if a student responds correctly to 3 or 4 items. Thus,

$$TCC = E(X|\theta) = 1 \times P(X=1|\theta) + 2 \times P(X=2|\theta). \quad (5)$$

Since a score of 1 corresponds to correctly responding to either one or two out of four items, the probability of obtaining a score of 1 corresponds to the probability of correctly responding to one out four items plus the probability of correctly responding to two out of four items. Let $P1(\theta)$ stand for the probability of getting one correct out of four, and let $Q_i(\theta)$ be the probability of an incorrect response. Then, modeling the item responses as conditionally independent, the equation for $P1(\theta)$ is given by:

$$P1(\theta) = P_1(\theta)\, Q_2(\theta)\, Q_3(\theta)\, Q_4(\theta) + Q_1(\theta)\, P_2(\theta)\, Q_3(\theta)\, Q_4(\theta) + Q_1(\theta)\, Q_2(\theta)\, P_3(\theta)\, Q_4(\theta) + Q_1(\theta)\, Q_2(\theta)\, Q_3(\theta)\, P_4(\theta). \quad (6)$$

Similarly, we can find the equation for responding correctly to two out of four items, $P2(\theta)$, as:

$$P2(\theta) = P_1(\theta)\, P_2(\theta)\, Q_3(\theta)\, Q_4(\theta) + P_1(\theta)\, Q_2(\theta)\, P_3(\theta)\, Q_4(\theta) + P_1(\theta)\, Q_2(\theta)\, Q_3(\theta)\, P_4(\theta) + Q_1(\theta)\, P_2(\theta)\, P_3(\theta)\, Q_4(\theta) + Q_1(\theta)\, P_2(\theta)\, Q_3(\theta)\, P_4(\theta) + Q_1(\theta)\, Q_2(\theta)\, P_3(\theta)\, P_4(\theta). \quad (7)$$

Thus,

$$P(X=1|\theta) = P1(\theta) + P2(\theta). \quad (8)$$

The equation for $P(X=2|\theta)$ is derived in a similar fashion. Since a score of 2 corresponds to correctly responding to either three or four out of four items, the probability of obtaining a score of 2 corresponds to the probability of correctly responding to three out of four items, $P3(\theta)$, plus the probability of correctly responding to all four items, $P4(\theta)$. Modeling the item responses as conditionally independent results in the following equations for $P3(\theta)$ and $P4(\theta)$:

$$P3(\theta) = P_1(\theta)\, P_2(\theta)\, P_3(\theta)\, Q_4(\theta) + P_1(\theta)\, P_2(\theta)\, Q_3(\theta)\, P_4(\theta) + P_1(\theta)\, Q_2(\theta)\, P_3(\theta)\, P_4(\theta) + Q_1(\theta)\, P_2(\theta)\, P_3(\theta)\, P_4(\theta), \quad (9)$$

and

$$P4(\theta) = P_1(\theta)\, P_2(\theta)\, P_3(\theta)\, P_4(\theta). \quad (10)$$

Thus,

$$P(X=2|\theta) = P3(\theta) + P4(\theta). \quad (11)$$

Substituting equations (8) and (11) into (5) gives the equation for the TCC.

Next we turn to the derivation of the TIF equation. The basic definition is the same in all cases:

$$I(\theta) = \{\frac{d}{d\theta}E[X(\theta)]\}^2 / Var[X(\theta)].$$

As noted above, $E[X(\theta)]$ is, by definition, the TCC; thus, we must calculate the derivative of the TCC with respect to θ. In this case, the TCC is the sum of 15 terms (four terms for P1, six for P2, four for P3, and one for P4), but each term is simply the product of conditional probabilities. Thus, taking the derivative of the TCC is again easily accomplished by applying the chain rule to each term.

Next we again consider the term in the denominator, $Var[X(\theta)]$. In general terms, this equation is given by:

$$Var[X(\theta)] = E[X^2(\theta)] - \{E[X(\theta)]\}^2 \quad (12)$$

The first term is defined as follows:

$$E[X^2(\theta)] = 1^2 \times P(X=1|\theta) + 2^2 \times P(X=2|\theta) \quad (13)$$

Substituting (8) and (11) into (13) and (5), and then substituting (13) and (5) into (12) yields the solution for $Var[X(\theta)]$. Thus, combining the solution for $Var[X(\theta)]$ with the solution for the derivative of the TCC with respect to θ yields the solution for $I(\theta)$ by substituting the results into equation (12). The final detailed TIF equation is omitted here to save space.

TCC and TIF for the five-item and six-item sets

The calculation of the TCC for the five-item and six-item sets proceeds in exactly the same way as for the four-item sets with the equation:

$$TCC = E(X|\theta) = 1 \times P(X=1|\theta) + 2 \times P(X=2|\theta).$$

However, now $P(X=1|\theta)$ and $P(X=2|\theta)$ must be defined relative to five-item and six-item sets. Specifically, the probability of a score of X=1 is equal to the sum of the probabilities of correctly responding to one out

five items and two out of five items for the five-item case; while it is equal to the sum of the probabilities of correctly responding to one out of six items and two out of six items for the six-item case. The equations for these probabilities are again constructed based on the assumption of conditional independence and generalize in the obvious way from the four-item case, naturally involving more terms. Similarly, the probability of a score of X=2 is defined for the five-item sets as the sum of the probabilities of correctly responding to 3, 4, and 5 items, while it is defined for the six-item sets as the sum of the probabilities of correctly responding to 3, 4, 5, and 6 items. Because these equations are longer and are the obvious generalizations of the equations for the case of 4 items, they are omitted here to save space.

Next we discuss the derivation of the TIF equation. The basic definition is, of course, the same as in all the cases:

$$I(\theta) = \{\frac{d}{d\theta}E[X(\theta)]\}^2 / Var[X(\theta)].$$

As in all the previous cases, this involves taking the derivative of the TCC with respect to θ. The TCC again consists of sums of terms that are products of conditional probabilities; so, the derivatives will again be easily calculated from applying the standard chain rule of elementary calculus, though the number of terms will be quite large. The calculation of Var[X(θ)] also proceeds in the same way as for the four-item sets in that equations (12) and (13) are both still applicable to the five- and six-item cases. As discussed in regard to the TCC equation, the only difference is that the equations for P(X=1|θ) and P(X=2|θ) must be expanded from the four-item case to the five-item and six-item cases.