Congruence and Similarity of Triangles
An ADE Mathematics Lesson
Days 21-30

Author	ADE Content Specialists
Grade Level	10th grade
Duration	Ten days

Aligns To

Mathematics HS:
Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representation
PO 1. Create and explain the need for equivalent forms of an equation or expression.

Strand 4: Geometry and Measurement
Concept 1: Geometric Shapes
PO 3. Create and analyze inductive and deductive arguments concerning geometric ideas and relationships.
PO 4. Apply properties, theorems, and constructions about parallel lines, perpendicular lines, and angles to prove theorems.
PO 6. Solve problems using angle and side length relationships and attributes of polygons.
PO 8. Prove similarity and congruence of triangles.
PO 10. Solve problems using right triangles, including special triangles.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.
PO 2. Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s).
PO 4. Generalize a solution strategy for a single problem to a class of related problems; explain the role of generalizations in inductive and deductive reasoning.
PO 5. Summarize and communicate mathematical ideas using formal and informal reasoning.
PO 7. Find structural similarities within different algebraic expressions and geometric figures.

Connects To

Mathematics HS:
Strand 1: Number and Operations
Concept 3: Estimation
PO 2. Use estimation to determine the reasonableness of a solution.

Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 2. Solve formulas for specified variables.

Strand 4: Geometry and Measurement
Concept 2: Transformation of Shapes
PO 3. Sketch and describe the properties of a 2-dimensional figure that is the result of two or more transformations.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 3. Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.

Overview

In this lesson, you will learn how to determine whether two triangles are similar or congruent and will be able justify their answer.

Purpose

It is often necessary to determine the relationship between two geometric figures to solve a problem. After this lesson, you will be able to determine the relationship between two given triangles to determine if they are similar, congruent, or neither to assist with further problem-solving in geometry.

Materials

- Similarity and Congruence worksheets
- Protractor
- Scissors
- Ruler

Objectives

Students will:

- Solve problems using angle and side length relationships and attributes of polygons.
- Prove similarity and congruence of triangles.
- Determine different methods to prove similarity and congruence of triangles.

Lesson Components

Prerequisite Skills: In 6^{th} through 9^{th} grades, you learned to recognize various attributes associated with triangles. You made determinations as to the nature of these triangles (right, isosceles, equilateral). You learned how to measure the sides and angles of various triangles and learned the Triangle-Sum Theorem and the Triangle-Inequality Theorem.

Vocabulary: *triangle, right triangle, isosceles triangle, equilateral triangle, scalene triangle, side of a triangle, angle of a triangle, vertex, hypotenuse, leg, similarity, congruence*

Session 1 (5 days)

1. Determine if two triangles are congruent by several different methods including physically measuring the triangles and using formulas for determining the congruence of triangles such as SSS, SAS, ASA, AAS, and H-L.

Session 2 (5 days)

1. Determine if two triangles are similar by several different methods including physically measuring the triangles and using formulas for determining the similarity of triangles such as SSS, SAS, and AA.

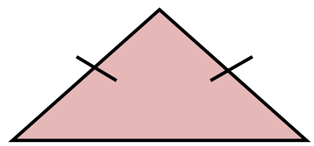
Assessment

There are two assessments that will help pinpoint misconceptions before moving on to more complex comparisons. The first assessment is focused on congruence of triangles and the second assessment is focused on similarity of triangles.

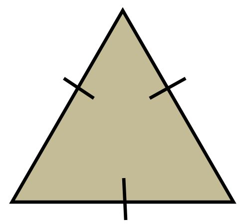
Congruence and Similarity of Triangles
Session 1 – Congruence

Review - Types of Triangles

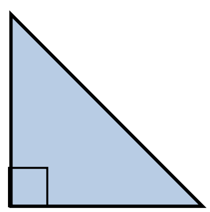
An **isosceles triangle** is a triangle that has two or more congruent sides (note: equilateral triangles are a subset of isosceles triangles).



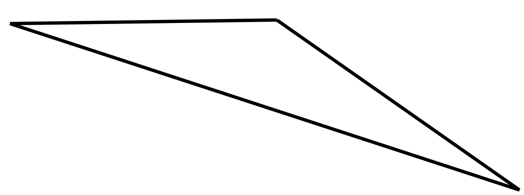
An **equilateral triangle** is a triangle in which all sides are congruent.



A **right triangle** is a triangle that contains a right angle.



A **scalene triangle** is a triangle with no congruent sides.



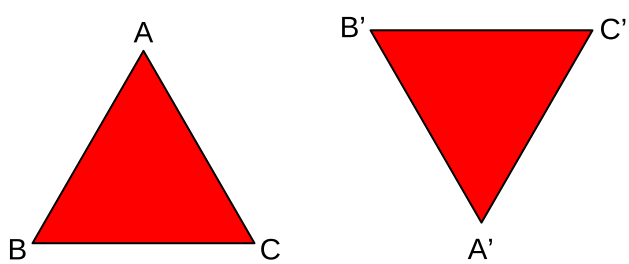
We can find the relationship of one triangle to another by making determinations of the size and shape of each triangle. Two triangles that have the same shape are said to be similar triangles. Two triangles that have the same size and shape are said to be congruent triangles. All congruent triangles are also similar triangles but not all similar triangles are congruent triangles.

Congruent triangles have the same shape and exactly the same size. We use the symbol $\cong$ to represent "is congruent to".

Similar triangles are triangles that have the same shape and are related in size by a scale factor. We use the symbol **~** to represent "is similar to".

Study the following examples of congruent triangles.

Example 1:



If $\Delta\ ABC \cong \Delta\ A'B'C'$, then the following relationships hold:

- $AB \cong A'B'$, $AC \cong A'C'$, $BC \cong B'C'$
- $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$
- $\angle BAC \cong \angle B'A'C'$, $\angle ABC \cong \angle A'B'C'$, $\angle ACB \cong \angle A'C'B'$

Example 2:

If two triangles are congruent and one triangle is an equilateral triangle, the other triangle must also be an equilateral triangle. Why would this be a true statement?

If two triangles are congruent and one triangle is an isosceles triangle, the other triangle must also be an isosceles triangle. Why would this be a true statement?

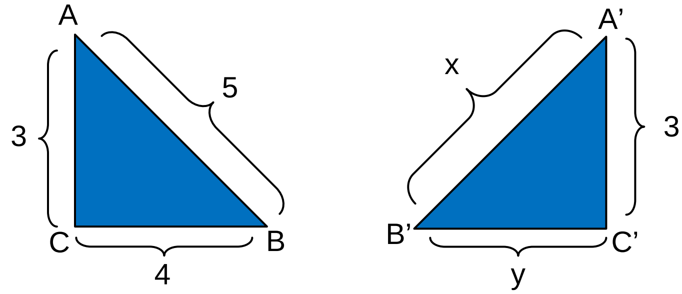
If two triangles are congruent and one triangle is a right triangle, the other triangle must also be a right triangle. Why would this be a true statement?

If two triangles are congruent and one triangle is a scalene triangle, the other triangle must also be a scalene triangle. Why would this be a true statement?

Can you summarize all the statements in the four parts of Example 2 into one or two succinct statements?

Example 3:

Fill in the measures of the missing sides, x and y, of the right triangle given that $\Delta\ ABC \cong \Delta\ A'B'C'$.

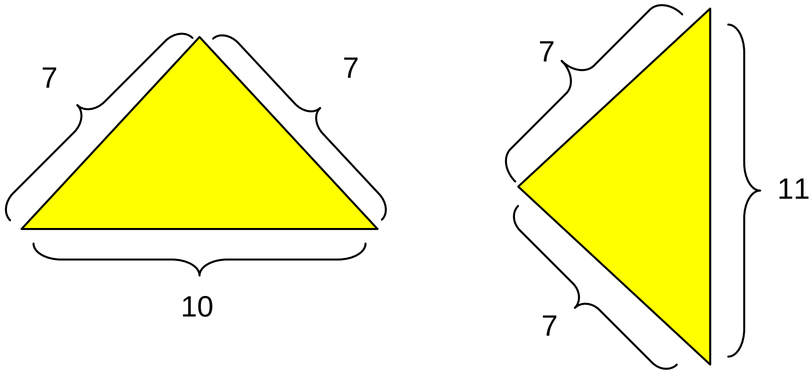


Example 4:

The measures of two angles of a triangle are 30° and 45°. The measures of a second triangle are also 30° and 45°. Are the two triangles congruent? Justify your answer.

Example 5:

Determine if the following two triangles are congruent. Justify your answer.



We can determine if two triangles are congruent by several different postulates.

Two triangles are congruent if each side of one triangle is congruent to each side of the other triangle. **(SSS)**

Two triangles are congruent if two sides of one triangle and the included angle of one triangle are congruent to two sides and the included angle of the other triangle. **(SAS)**

Two triangles are congruent if two angles of one triangle and the included side of one triangle are congruent to two angles of one triangle and the included side of the other triangle. **(ASA)**

Two triangles are congruent it two angles of one triangle and the non-included side are congruent to two angles of another triangle and the non-included side. **(AAS)**

Two right triangles are congruent if the hypotenuse and leg of one triangle are congruent to the hypotenuse and leg of the other triangle. **(H-L)**

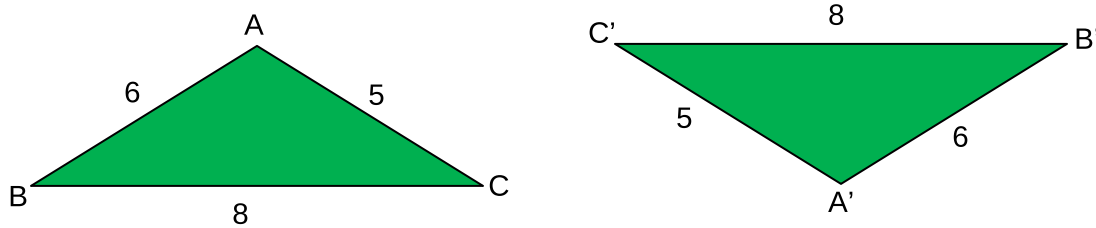
We can also make statements about the angles and sides of congruent triangles.

Corresponding **p**arts of **c**ongruent **t**riangles are **c**ongruent **(CPCTC)**

(When triangles are congruent, each corresponding side has the same measure and each corresponding angle has the same measure.)

Example 6:

Look at the following triangles and determine if they are congruent by one of the congruence postulates. Justify your answer. What statements can you make about the angles of the two triangles?

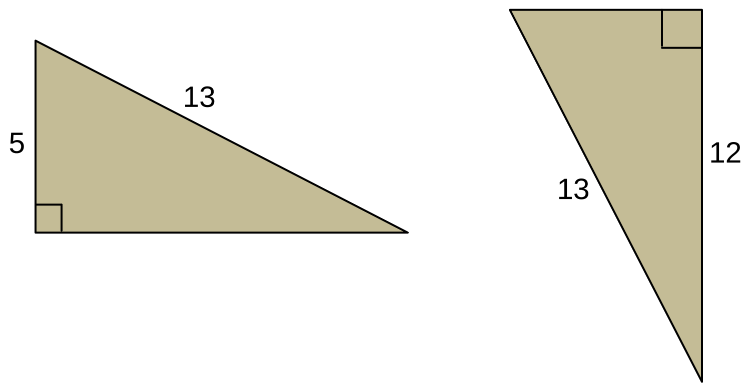


Example 7:

Are all 30°-60°-90° triangles congruent? Justify your answer.

Example 8:

Look at the following triangles and determine if they are congruent by one of the congruence postulates. The hypotenuse of each triangle is 13. The shortest leg in the first triangle is 5 and the longest leg of the second triangle is 12. Justify your answer. What statements can you make about the missing sides and angles of the two triangles?



Make up a congruence problem based on one of the congruence postulates. Show the problem in the space provided. Show at least one correct method to solve the problem you created.

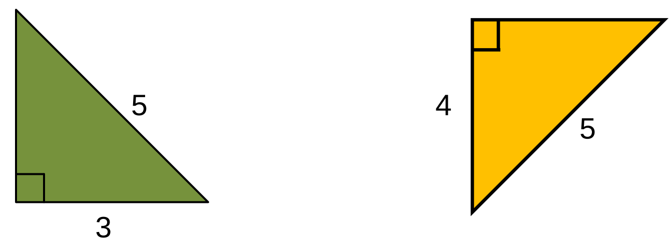
Congruence and Similarity of Triangles
Assessment 1 – Congruence

Write out the meaning of each congruence postulate in the space provided.

Postulate	Meaning
SAS	
AAS	
SSS	
SAS	
H-L	
CPCTC	

Determine if the following sets of triangles are congruent. Justify your answer.

Problem 1:



The hypotenuse of each triangle is 5. The shortest leg in the first triangle is 3 and the longest leg of the second triangle is 4.

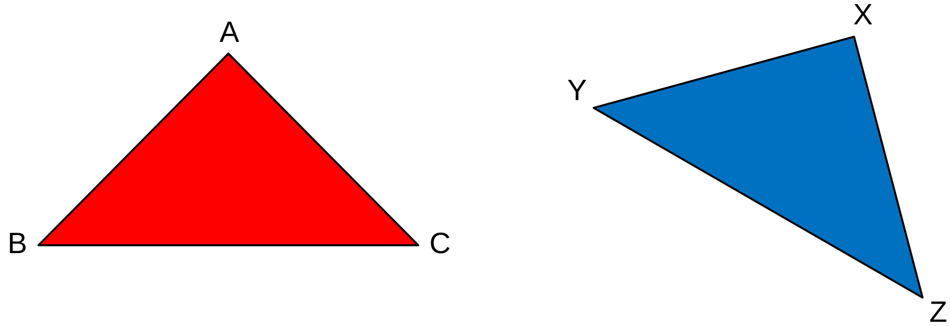
Problem 2:

The shortest sides of two triangles are 4 and 5, respectively. The angle in between the two sides of the first triangle is 48 degrees and the angle in between the two sides of the second triangle is 42 degrees. Which is a true statement about the relationship of the two triangles? Justify your answer.

1. The two triangles are congruent by SAS.
2. The two triangles can never be congruent because the angle in between the two given sides is different.
3. There is not enough information to determine if the two triangles are congruent.

Problem 3:

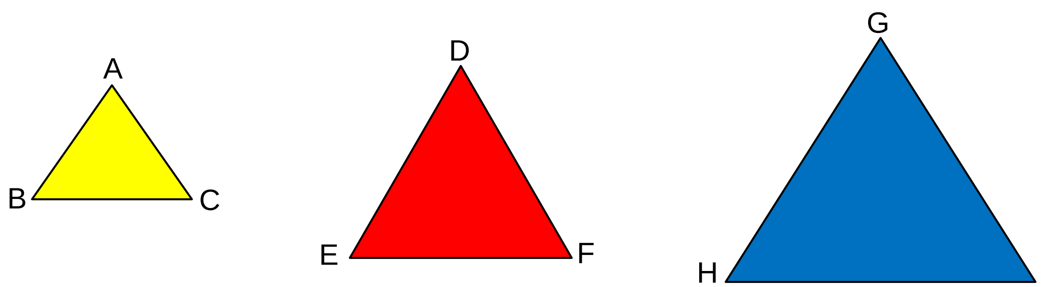
If $AB = 4$, $BC = 8$, $\angle C = 40°$ and $YZ = 8$, $XY = 4$ and $\angle Y = 40°$, is $\Delta ABC \cong \Delta XYZ$? Why or why not?



Congruence and Similarity of Triangles
Session 2 – Similarity

Sometimes, one triangle has exactly the same shape as another triangle but is not the same size. Consider the case of equilateral triangles. In equilateral triangles, all of the triangles have exactly the same shape, because all angles are 60°. All sides of an equilateral triangle are the same. Are all equilateral triangles then similar? Remember that similar triangles are triangles that have the same shape and are related in size by a scale factor. We use the symbol **~** to represent "is similar to".

Consider the following set of equilateral triangles.



Example 1:

In the yellow triangle, ΔABC, all sides are 2. In the red triangle, ΔDEF , all sides are 3. In the blue triangle, ΔGHI , all sides are 5. Are any of the triangles similar to the others in the set?

Solution:

Compare the yellow and red triangles. Each corresponding side of the yellow triangle, ΔABC, has a ratio of 2:3 to the corresponding side of the red triangle, ΔDEF . In other words, each side of ΔABC is $\frac{2}{3}$ of each side of ΔDEF .

Therefore, we know that the yellow triangle is similar to the red triangle and we write that

$\Delta ABC \sim \Delta DEF$ **.**

Compare the red and blue triangles. Each corresponding side of the red triangle, ΔDEF, has a ratio of 3:5 to the corresponding side of the blue triangle, ΔGHI. In other words, each side of ΔDEF is $\frac{3}{5}$ of each side of ΔGHI.

Therefore, we know that the red triangle is similar to the blue triangle and we write that $\Delta DEF \sim \Delta GHI$.

Is the yellow triangle similar to the blue triangle?

Following the same logic used to compare ΔABC with ΔDEF and ΔDEF with ΔGHI, we can compare ΔABC with ΔGHI.

The sides of ΔABC are each 2 units. The sides of ΔGHI are each 5 units. Therefore each corresponding side of ΔABC is $\frac{2}{5}$ of the corresponding side of ΔGHI. We know that the ratio of the corresponding sides of the yellow triangle to the corresponding sides of the blue triangle is 2:5.

Therefore, we know that the yellow triangle is similar to the blue triangle and we write that $\Delta ABC \sim \Delta GHI$.

The similarity postulate is known as **SSS** and states that if three sides of a triangle are in the same ratio to three corresponding sides of another triangle, the two triangles are similar.

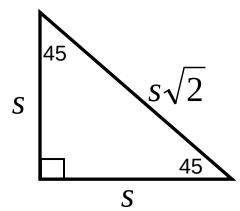
Example 2:

Are special right triangles similar to each other? Examine both 45°-45°-90° triangles and 30°-60°-90° triangles.

Solution:

Case 1: 45°-45°-90° triangles

What do we know about the special relationships in 45°-45°-90° triangles?



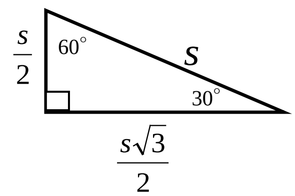
We know that the relationship of each leg of the right triangle to the hypotenuse is

s : $s\sqrt{2}$ or 1: $1\sqrt{2}$ This means that the hypotenuse of a right triangle is $\sqrt{2}$ times the leg. This ratio is constant no matter what the measure of the hypotenuse is. Therefore, we know that in 45°-45°-90° triangles, all angles are the same and that corresponding sides have the same ratio.

Therefore all 45°-45°-90° triangles are similar.

Case 2: 30°-60°-90° triangles

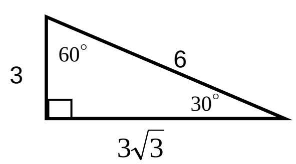
What do we know about the special relationships in 30°-60°-90° triangles?

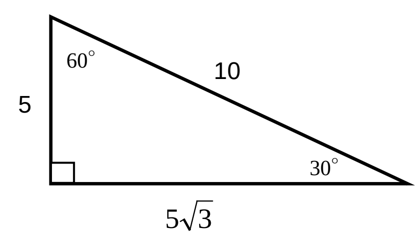


If we compare the side of one 30°-60°-90° triangle to the side of the corresponding side of another triangle, do we obtain the same ratio?

We will not spend time in this lesson proving the special relationships in these special right triangles. We will talk about the necessary conditions for the two 30°-60°-90° triangles to be similar and if they are met. If the hypotenuse of the triangle is *s*, then the side opposite the 30° angle will always be half of the measure of the hypotenuse. The side opposite the 60° angle will always be one half of the measure of the hypotenuse times $\sqrt{3}$.

Let's look at an example of two different 30°-60°-90° triangles.





The ratios of the corresponding sides of the two triangles are $\frac{3}{5} = \frac{3\sqrt{3}}{5\sqrt{3}} = \frac{6}{10}$.

These ratios all reduce to $\frac{3}{5}$. Since the ratios of the corresponding sides are the same, the two special right triangles are similar by SSS. However, this is just an example and does not prove that all 30°-60°-90° triangles are similar. We can follow the same process though and show that the ratios of the corresponding sides of these triangles will always be the same and therefore all 30°-60°-90° will be similar.

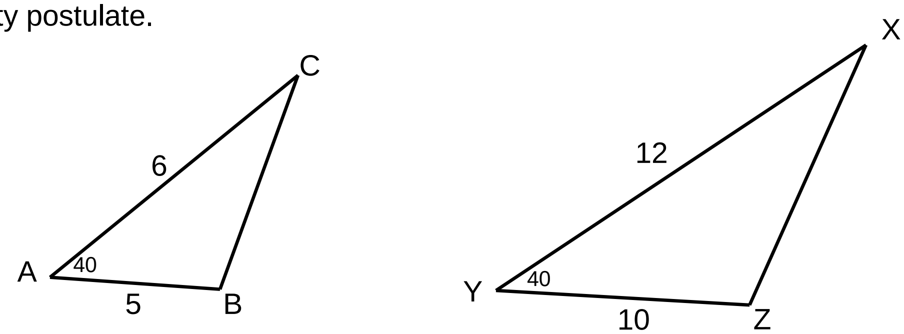
Show one example of two 45°-45°-90° triangles and one example of two 30°-60°-90° triangles that are similar. Justify your work.

45°-45°-90° triangle
30°-60°-90° triangle

The similarity postulate **SAS** states that if two sides and the included angle of one triangle are in the same ratio as the two corresponding sides and the included angle of the other triangle, the two triangles are similar.

Example 2:

Examine the two triangles below and determine if ΔABC is similar to ΔXYZ by the SAS similarity postulate.



The SAS triangle states that the ratio of two corresponding sides must be in the same ratio and included angle between the corresponding sides must be the same.

Note that $\frac{6}{12} = \frac{5}{10}$ since both reduce to $\frac{1}{2}$.

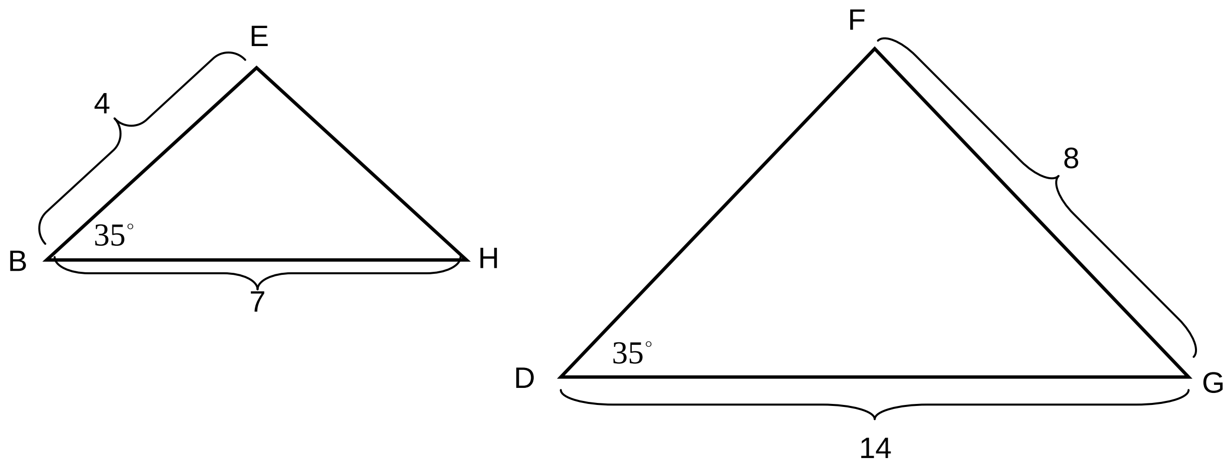
(Can also be written as: $\frac{5}{6} = \frac{10}{12}$ since both reduce to $\frac{5}{6}$)

$$m\angle A = m\angle C = 40°.$$

Therefore ΔABC ~ ΔXYZ by SAS.

Example 3:

Examine the two triangles below and determine if ΔBEH is similar to ΔDFG.



Solution:

We know that $\angle EBH = \angle FDG = 35°$.

The ratio of side BH to side DG is 7:14 or 1:2.

The ratio of side BE to side FG is 4 to 8 or 1:2.

However, $\angle EBH$ is the included angle between sides EB and BH.

There is no information about $\angle FGD$ which is the included angle between sides FG and DG.

Therefore, we cannot use **SAS** to determine that the triangles are similar.

There is no information about the measures of the third sides, EH and FD. Since neither triangle is a special right triangle or an equilateral triangle, we cannot make any assumptions about the measure of the third sides.

Therefore, we have insufficient information to say that the two triangles are similar by **SSS**.

There is insufficient information to prove that $\Delta BEH \sim \Delta DFG$.

The similarity postulate **AA** states that if two angles of one triangle are equal to two angles of another triangle, the two triangles are similar.

If two angles of a triangle are equal to two angles of another triangle, then the third angles of each triangle must also be equal. Can you explain the reason for this?

Example 4:

One triangle has two angles that measure 44° and 55°. A second triangle has two angles that measure 44° and 81°. Are the two triangles similar? Explain your reasoning.

Solution:

The sum of the angles of a triangle total 180°. In the first triangle, the third angle measures $180° - (44° + 55°) = 81°$. Since two angles of the first triangle are the same as two angles of the second triangle, the triangles must be similar by the AA Similarity Postulate.

Example 5:

One angle of a right triangle is 37.5°. One angle of a second right triangle measures 52.5°. Are the two triangles similar?

Solution:

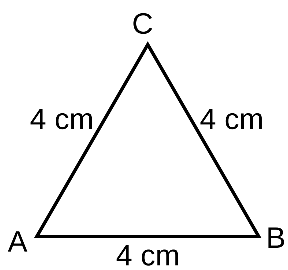
In a right triangle, the two acute angles are complementary. Since 37.5° + 52.5° = 90°, the two triangles have two acute angles that must be the same. All angles of each triangle are the same (37.5°, 52.5°, and 90°).

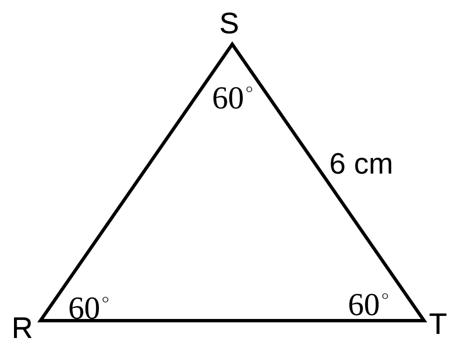
Since two angles of the first triangle are the same as two angles of the second triangle, the triangles must be similar by the AA Similarity Postulate.

Are right triangles always similar if one acute angle of one right triangle is the same as one acute angle of another right triangle? Justify your answer.

Determine if the following sets of triangles are similar. Justify your answer in the space provided.

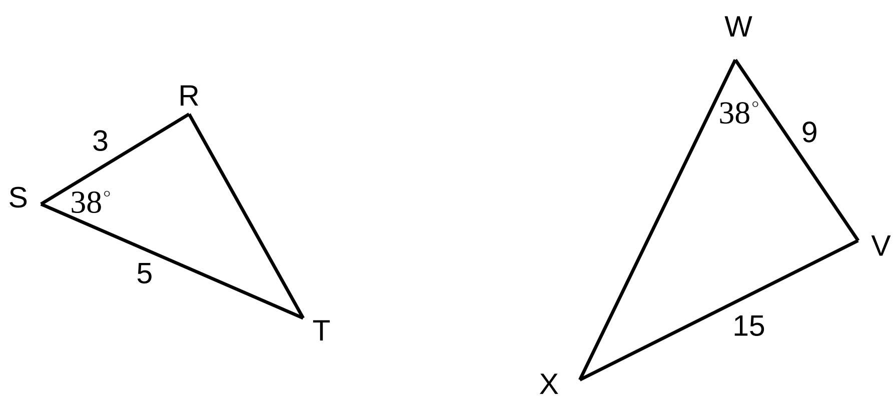
1. Each side of ΔABC is 4 cm. In ΔRST, the measure of each angle is 60° and one side is 6 cm. Determine if the two triangles are similar and justify your answer in the space provided.





2. The acute angle of one right triangle is 34°. The acute angle of a second right triangle is 56°. Are the two right triangles similar? Justify your answer.

3. Are the two triangles ΔRST and ΔVWX shown below similar? Justify your answer.



4. Complete the following table showing two triangles with measurements marked on the triangles that make the triangles similar by the given similarity postulate.

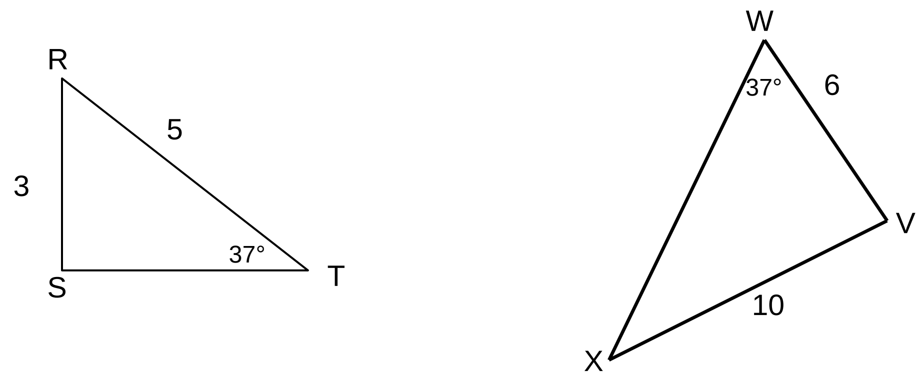
SIMILARITY POSTULATE	EXAMPLE
SSS	
SAS	
AA	

Congruence and Similarity of Triangles
Assessment 2 – Similarity

1. Are two equilateral triangles always similar? Justify your answer.

2. The acute angle of one right triangle is 44°. The acute angle of a second right triangle is 46°. Are the two right triangles similar? Justify your answer.

3. Are the two triangles ΔRST and ΔVWX shown below similar? Justify your answer.



4. Describe the similarity postulates in your own words.

Side-side-side (SSS)

Side-angle-side (SAS)

Angle-angle (AA)

Extensions

1. Examine similar and congruent triangles in architecture.

2. Examine similar and congruent triangles in art.

3. Explore tests used to determine if polygons (other than triangles) are congruent.
   http://www.mathopenref.com/congruentpolygons.html
   http://www.mathopenref.com/congruentpolygonstests.html
   http://www.mathopenref.com/congruenttriangles.html

Sources

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