Arizona Department of Education
8-Week Pandemic Curriculum

Mathematics
Grade 10

This curriculum was designed for self-guided student learning and aligns to Arizona Academic Standards.

Developed February, 2010

Curriculum Overview
Mathematics Grade 10

Lesson Title	Duration	Materials	Assessments (number)
Surface Area and Volume	**Days 1-10** Session 1 (6 days) Session 2 (4 days)	• surface area worksheets (5) • volume worksheets (3) • ruler • modeling dough • graph paper or other lined paper • unlined paper • 2 enrichment activities each with several worksheets	3
Polygons	**Days 11-15** Session 1 (2 days) Session 2 (3 days)	• polygon worksheets (5) • quadrilateral worksheets (5) • lined paper such as graph paper • ruler • compass • protractor	2
Geometric Probability	**Days 16-20** Session 1 (2 days) Session 2 (3 days)	• geometric probability worksheets • session one (1) • session two (4) • ruler • enrichment worksheet	1
Congruence and Similarity of Triangles	**Days 21-30** Session 1 (5 days) Session 2 (5 days)	• congruence worksheets (5) • similarity worksheets (6)	3
Circle Relationships	**Days 31-40** Session 1 (5 days) Session 2 (5 days)	• circle relationship worksheets • compass • ruler/protractor	2

Surface Area and Volume
An ADE Mathematics Lesson
Days 1-10

Author	ADE Content Specialists
Grade Level	10th grade
Duration	Ten days

Aligns To

Mathematics HS:
Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 1. Create and explain the need for equivalent forms of an equation or expression.

Strand 4: Geometry and Measurement
Concept 1: Geometric Shapes
PO 1. Use the basic properties of a circle (relationships between angles, radii, intercepted arcs, chords, tangents, and secants) to prove basic theorems and solve problems.
PO 2. Visualize solids and surfaces in 3-dimensional space when given 2-dimensional representations and create 2-dimensional representations for the surfaces of 3-dimensional objects.
PO 3. Create and analyze inductive and deductive arguments concerning geometric ideas and relationships.
PO 6. Solve problems using angle and side length relationships and attributes of polygons.
Concept 4: Measurement
PO 5. Calculate the surface area and volume of 3-dimensional figures and solve for missing measures.

Strand 5: Structure and Logic
Concept 1: Algorithms and Algorithmic Thinking
PO 1. Select an algorithm that explains a particular mathematical process; determine the purpose of a simple mathematical algorithm.

Connects To

Mathematics HS:
Strand 1: Number and Operations
Concept 3: Estimation
PO 2. Use estimation to determine the reasonableness of a solution.

Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 2. Solve formulas for specified variables.

Strand 4: Geometry and Measurement
Concept 1: Geometric Shapes
PO 4. Apply properties, theorems, and constructions about parallel lines, perpendicular lines, and angles to prove theorems.
PO 7. Use the hierarchy of quadrilaterals in deductive reasoning.
Concept 4: Measurement
PO 1. Use dimensional analysis to keep track of units of measure when converting.
PO 3. Determine the effect that changing dimensions has on the perimeter, area, or volume of a figure.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 3. Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.

Aligns To	Connects To
Mathematics HS: **Concept 2: Logic, Reasoning, Problem Solving, and Proof** **PO 1.** Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made. **PO 2.** Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s). **PO 4.** Generalize a solution strategy for a single problem to a class of related problems; explain the role of generalizations in inductive and deductive reasoning. **PO 5.** Summarize and communicate mathematical ideas using formal and informal reasoning. **PO 7.** Find structural similarities within different algebraic expressions and geometric figures.	

Overview

It is often necessary to find the surface area or volume of 3-D objects. Surface area is easier if you conceptually understand what finding the surface area of an object means. You will recognize the similarities and difference between finding surface areas and volumes of prisms and pyramids.

Purpose

Finding the surface area and volume of 3-D objects allows you to solve problems in which you must find either the capacity of a 3-D object or the amount of actual surface area that a 3-D object contains.

Materials

- Surface Area and Volume worksheets
- Ruler
- Play-doh® or modeling dough
- Recipes for Modeling Dough handout
- Graph paper or other lined paper
- Unlined paper

Objectives

Students will:

- Find the surface area of prisms and pyramids.
- Find the volume of prisms and pyramids.
- Develop the formulas to find the surface areas and volumes of prisms and pyramids.
- Solve contextual problems that involve prisms and pyramids.

Lesson Components

Prerequisite Skills: In 7th and 8th grade mathematics, you learned to find area of two-dimensional objects. You will extend this concept to find the surface area of three-dimensional objects. You found volumes of very simple three-dimensional objects such as rectangular prisms. You will expand this to find the volumes of more complex three-dimensional objects. You need to be familiar with the various types of polygons to be able to name three-dimensional objects.

Vocabulary: *area, surface area, polyhedron, volume, prism, pyramid, base, height, slant height, rectangular prism, polygon, cylinder, cube, lateral surface area, total surface area, triangular prism, triangular pyramid, pentagonal prism, pentagonal pyramid, hexagonal prism, hexagonal pyramid*

Session 1 (6 days)

1. Find the lateral and total surface area of different forms of prisms.
2. Find the lateral and total surface area of different forms of pyramids.

Session 2 (4 days)

1. Find the volumes of different forms of prisms.
2. Find the volumes of different forms of pyramids.

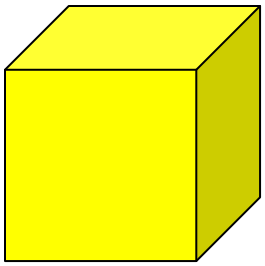
Assessment

There are three assessments for this lesson, one assessment after Session 1 and two assessments after Session 2. You will be able to pinpoint your errors and misconceptions before moving on to more complicated problems using surface area and volume.

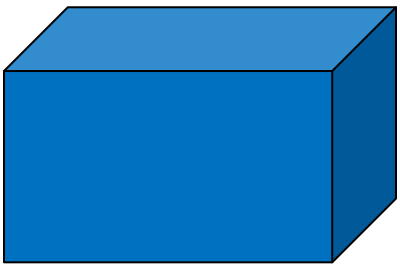
Surface Area
Session 1

To find the surface area of a three-dimensional object, it is important to recognize the names of each object. A **prism** is a three-dimensional figure called a polyhedron made up of two parallel congruent faces and lateral faces that are parallelograms. A prism is named for its base. Examples of three-dimensional objects that have two bases are shown below.

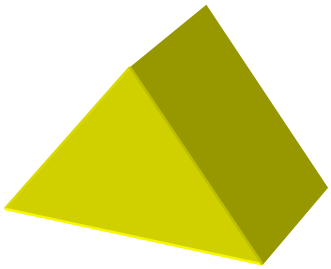
A prism with square bases is called a **cube**.



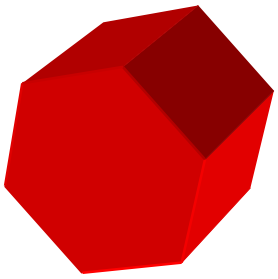
A prism with rectangles as bases is called a **rectangular prism**.



A prism with triangles as bases is called a **triangular prism**.



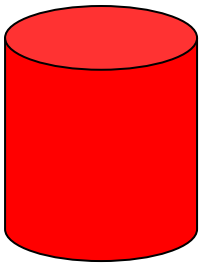
A prism with hexagons as bases is called a **hexagonal prism.**



A **pentagonal prism** would be a prism that has pentagons as its bases.

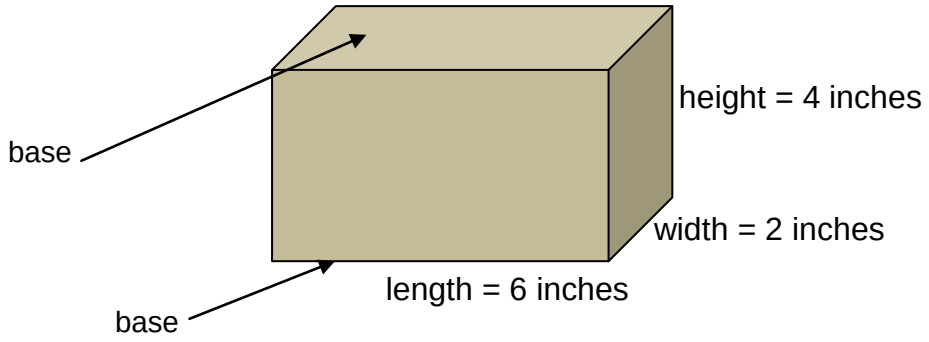
An **octagonal prism** would be a prism that has octagons as its bases.

A **cylinder** is a 3-dimensional figure composed of two congruent and parallel circular regions joined by a curved surface.



- To find the **lateral surface area** of a prism, find the area of each of the faces of the prism excluding the bases. Add the areas of these faces together to find lateral surface area.
- To find the **total surface area** of a prism, add the areas of the two bases to the lateral surface area.
- Remember that surface area is always in square units.

Example 1: Find the lateral surface area and the total surface area of the rectangular prism shown below.



Solution:

To find the <u>lateral surface area</u>, find the area of the four faces. Two of these faces have an area of 4 inches x 2 inches. Two of these faces have an area of 4 inches by 6 inches.

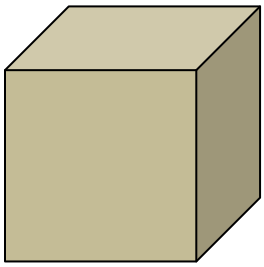
$$2(4 \bullet 2) + 2(6 \bullet 4) = 64$$

Therefore the lateral surface area of this rectangular prism is 64 square inches.

To find the <u>total surface area</u>, find the area of the two bases and add this on to the lateral surface area. Each base is 6 inches x 2 inches. The area of each base is 12 square inches. The area of both bases would be 24 square inches. Add this to 64 square inches. The total surface area of the rectangular prism is 88 square inches.

Example 2:

Find the lateral surface area and the total surface area of the cube shown below. Each side of the cube is 5 centimeters.



Solution:

The lateral surface area of the cube is 4 times one face as each face of a cube is identical. Since each edge is 5 centimeters, the area of each face is $5^2 = 25$ square centimeters. The lateral surface area is 4 x 25 square centimeters or 100 square centimeters.

The total surface area of the cube is the lateral surface area plus the surface area of the bases. Each base is 25 square centimeters. The total surface area is 100 square centimeters plus the area of the two bases which is 50 square centimeters. The area of the cube is 150 square centimeters.

We could have easily found that the total surface area of the cube was 150 square centimeters by multiplying the area of one face of the cube by six. 6(25) = 150 sq. cm.

It is very helpful to make a three dimensional object out of modeling dough to calculate the surface area. Make a rectangular prism out of one color of modeling dough and a cube out of another color. Do not make the cube and rectangular prisms too small or it will be difficult to measure them. Draw each of these in the boxes provided. Measure the length, width, and height of the rectangular prism and indicate this on your diagram. Measure the edge of the cube and indicate this on your drawing. Find the lateral surface area and total surface area of the rectangular prism and cube showing all work.

DIAGRAM OF RECTANGULAR PRISM

Lateral Surface Area

Total Surface Area

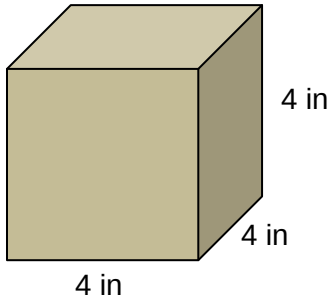
DIAGRAM OF CUBE

Lateral Surface Area

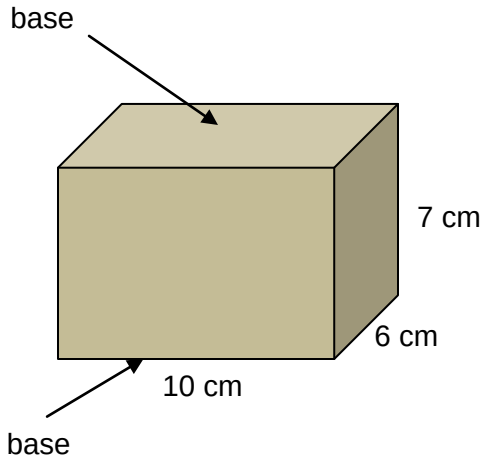
Total Surface Area

Find the lateral surface area and total surface area of each figure shown.

Cube with edges of 4 inches



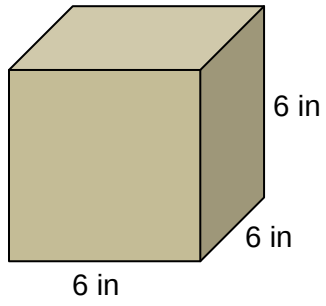
Rectangular prism with length of 10 cm, width of 6 cm, and height of 7 cm.



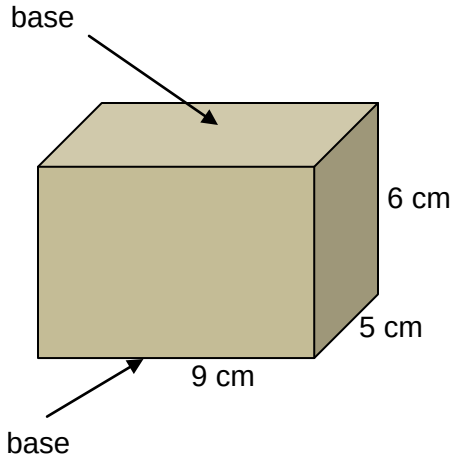
Surface Area
Assessment 1

Find the lateral surface area and total surface area of each figure shown. Show all your work in the space provided.

Cube with edges of 6 inches



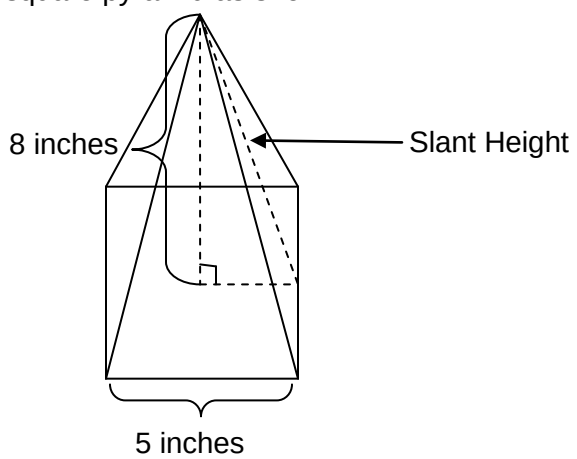
Rectangular prism with length of 9 cm, width of 5 cm, and height of 6 cm.



A **pyramid** is a three-dimensional figure whose base is a polygon and whose lateral faces are triangles that share a common vertex. A pyramid is named by its base. A triangular pyramid has a base that is a triangle. A square pyramid has a base that is a square. A hexagonal pyramid has a base that is a hexagon.

Example 1:

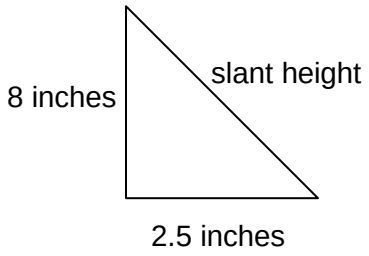
Find the total surface area of the square pyramid as shown.



Solution:

To find the surface area of a pyramid, it is necessary to find the slant height of the pyramid. Consider the square pyramid. The base of the pyramid is a square with a side of 5 inches. Therefore the area of the base would be 5 inches x 5 inches or 25 square inches. Each of the other four sides is a triangle. It is necessary to find the area of one of the triangles and then multiply this by four to obtain the area of all four triangles.

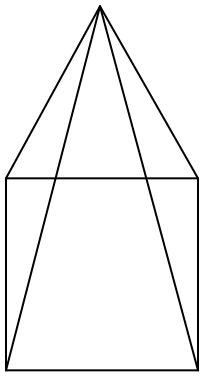
Look at the triangle indicated by dotted lines in the pyramid. The dimensions of the triangle are shown in the drawing below. Use the Pythagorean Theorem to find the slant height.



$$c^2 = a^2 + b^2 \Rightarrow c^2 = 8^2 + 2.5^2 = 70.25 \Rightarrow c \approx 8.4$$

The slant height is approximately 8.4 inches.

The area of each triangular face is given by



$$A = \frac{1}{2} b \bullet h$$

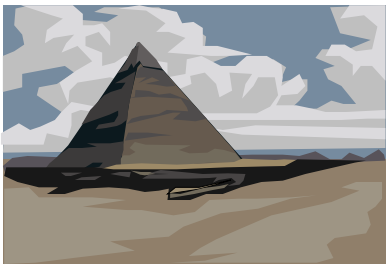
$$A = \frac{1}{2} b \bullet h \;\Rightarrow\; A = \frac{1}{2} \bullet 5 \bullet 8.4 \approx 21 \text{ square inches}$$

There are four of these triangles so that the lateral surface area is 4 x 21 square inches or 84 square inches. Add the area of the base onto the lateral surface area to obtain the total surface area of the square pyramid.

Total surface area ≈ 25 + 84 square inches ≈ 109 square inches.

Example 2

Find the lateral surface area and total surface area of the Great Pyramid of Giza whose base is a square. The length of one of the bases is 230 meters and its original height was 146 meters.

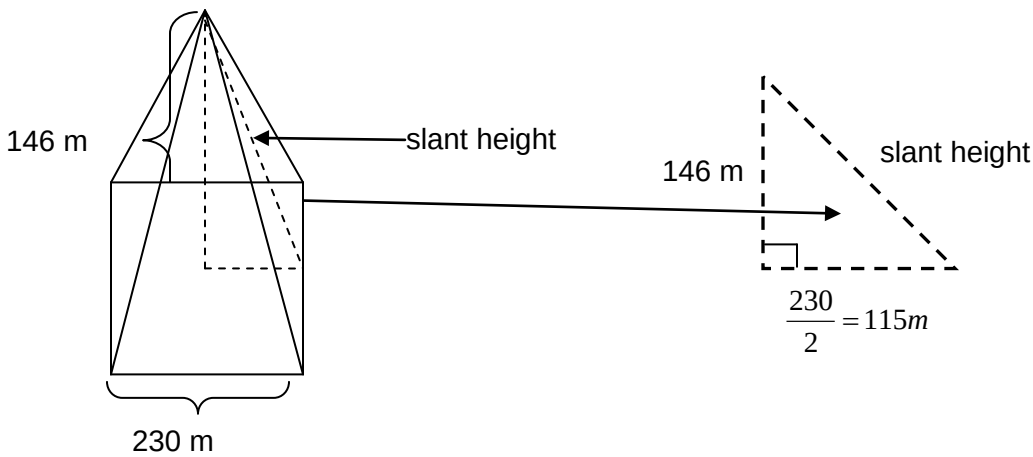


Solution:

The area of the base is a square whose side is 230 meters.

$$230m \bullet 230m = 52{,}900m^2$$

In order to find the area of each of the triangular faces, it is necessary to find the slant height of the pyramid. Use the Pythagorean Theorem to find this slant height.



$$c^2 = a^2 + b^2 \Rightarrow c^2 = 146^2 + 115^2 = 34{,}541 \Rightarrow c \approx 185.9\ m$$

The area of each of the four triangular sides is given by

$$A = \frac{1}{2} b \bullet h \ \Rightarrow\ A = \frac{1}{2} \bullet 230m \bullet 185.9m = 21{,}378.5\ m^2$$

The area of the four triangular sides is $4 \bullet 21{,}369m^2 = 85{,}516m^2$.

Therefore the lateral surface area of the pyramid is $85{,}516\ m^2$.

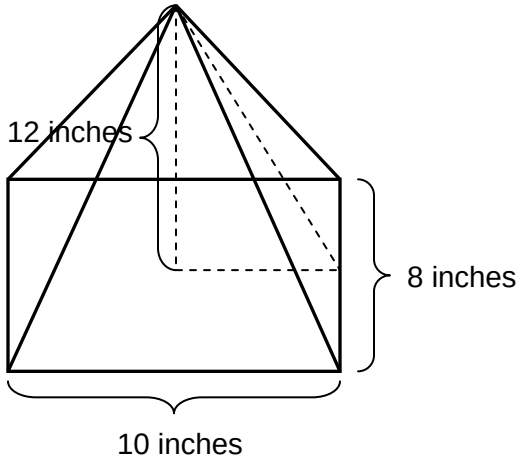
The total surface area of the pyramid is $52{,}900\ m^2 + 85{,}516\ m^2 \approx 138{,}416\ m^2$.

Problem 1:

Find the lateral and total surface area of a square pyramid whose base is 6 feet and whose height is 4 feet. Draw the pyramid and show all your work in the space provided.

Problem 2:

Consider the rectangular pyramid below. Find both its lateral and total surface area. Show all your work in the space provided. (Note – You may make this figure with modeling dough to calculate the surface area.)

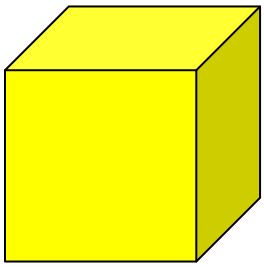


Volume
Session 2

- To find the volume of a prism, find the area of the base of the figure, and multiply this by its height. $V = Bh$ where *B* is the area of the base.
- To find the volume of a pyramid, find the area of the base of the figure, and multiply by its height. Then take one-third of this answer to find the volume. $V = \frac{1}{3}Bh$ where *B* is the area of the base.
- The units for volume are always in cubic units.

Example 1: CUBE

Find the volume of a cube whose edge is 5 cm.



To find the volume of a cube, multiply the area of the base ($A = s^2$) by the height of the cube. Since a cube has all three dimensions the same, the cube's height will also be s. Therefore, the volume of the cube will be $V = s^3$ which is often represented by $V = e^3$ where e stands for the edge of the cube which is the same as the side of the cube.

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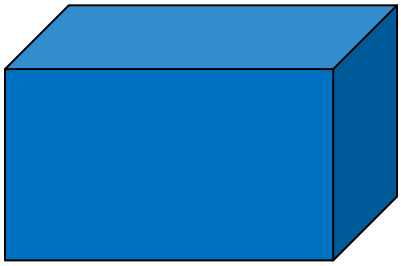
Solution: $V = e^3 \Rightarrow V = 5^3 = 125cm^3$. The volume of the cube is $125cm^3$.

Example 2: RECTANGULAR PRISM

Find the volume of a rectangular prism whose width is 5 inches, length is 7 inches, and height is 10 inches.



To find the volume of a rectangular prism, multiply the area of the base ($A = lw$) by the height of the rectangular prism. The volume of the rectangular prism will be:

$$V = Bh \Rightarrow V = lwh$$

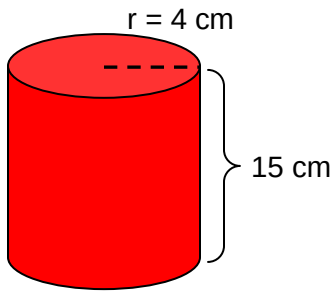
Solution:

$$V = Bh \Rightarrow V = lwh \quad V = 5inches \bullet 7inches \bullet 10inches = 350in^3$$

$$V = 350 \ \ in^3$$

Example 3: CYLINDER

Find the volume of a cylinder whose height is 15 cm and which has a radius of 4 cm. Express the answer in terms of π and in approximate units.



To find the volume of a cylinder, find the area of the base and multiply it by the height of the cylinder. Since the base of a cylinder is a circle, the area of the base can be found using the formula, $A = \pi \bullet r^2$. To find the volume of the cylinder, multiply this by the height.

$V = Bh \Rightarrow V = \pi \bullet r^2 \bullet h$.

Solution:

$$V = Bh \Rightarrow V = \pi \bullet r^2 \bullet h$$

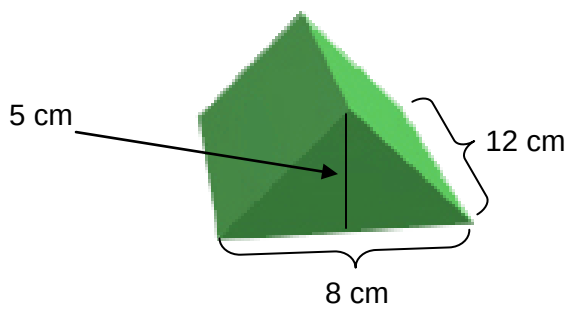
$$V = \pi \bullet 4^2 \bullet 15\ cm^3$$

$$V = \pi \bullet 16 \bullet 15 = 240\pi\ cm^3$$

$$V = 240\pi \approx 753.98\ cm^3$$

Example 4: TRIANGULAR PRISM

Find the volume of a triangular prism whose dimensions are as follows: base = 8 cm, height of base (h_1) = 5 cm, height of prism (h_2) = 12 cm. The base is a triangle so use the formula for calculating the area of a triangle times the height of the prism to determine the volume of the prism.

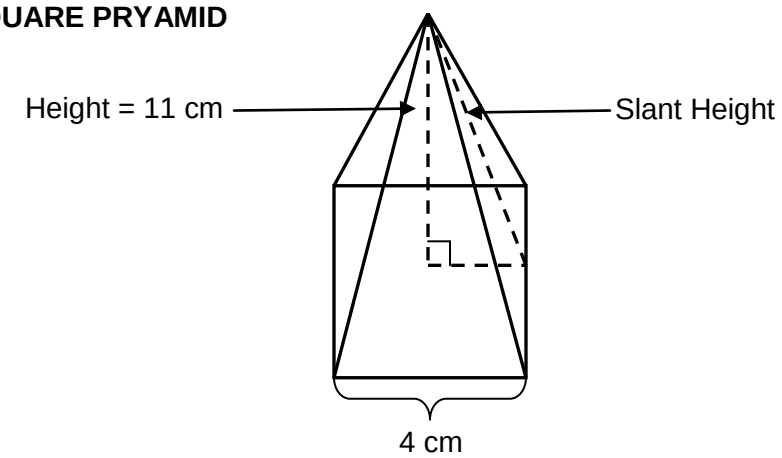


$$V = Bh \Rightarrow V = \frac{1}{2} \bullet b \bullet h_1 \bullet h_2 \quad \Rightarrow V = \frac{1}{2} \bullet 8 \bullet 5 \bullet 12 \Rightarrow V = 240\ cm^3$$

There is a relationship between the volume of a prism and the volume of a pyramid. The volume of the pyramid is 1/3 the volume of the prism that has the same base and height. The volume of a pyramid is equal to 1/3 the area of its base times its height.

$$V = \frac{1}{3} B \bullet h$$

Example 5: SQUARE PRYAMID



Find the volume of the square pyramid above whose height is 11 cm and which has a side of 4 cm for its square base. Can you find the slant height of this pyramid?

$$V = \frac{1}{3}B \bullet h \Rightarrow V = \frac{1}{3} \bullet 4^2 \bullet 11 = 58.67\ cm^3$$

Note that the height makes a right triangle with half of the side of the square and the slant height.

$$c^2 = a^2 + b^2 \Rightarrow c^2 = 2^2 + 11^2 \Rightarrow c^2 = 4 + 121 = 125 \Rightarrow c = \sqrt{125} \approx 11.18\ cm$$

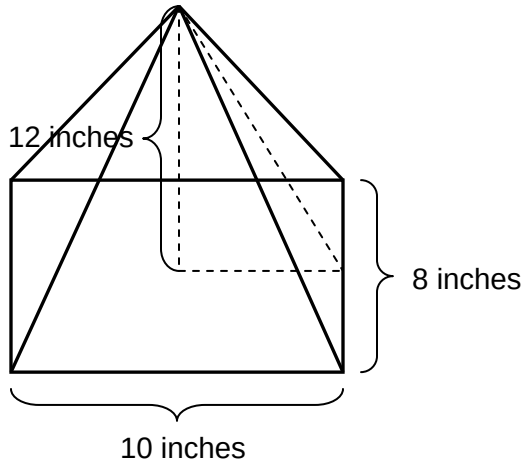
The slant height is approximately 11.18 cm.

Problem 1:

Find the volume of a square pyramid whose base is 6 feet and whose height is 4 feet. Draw the pyramid and show all your work in the space provided.

Problem 2:

Find the volume of the rectangular pyramid shown below. Show all your work in the space provided. (Note – you may make this figure with modeling dough to calculate the volume.)



Problem 3: Find the volume of a cube whose edge is 3 inches. Draw the cube and show your work in the space provided.

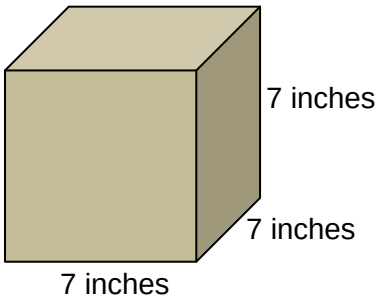
Problem 4: CYLINDER

Find the volume of a cylinder whose height is 4 in and which has a radius of 2 in. Express the answer in terms of π and in approximate terms. Draw the cylinder and show your work in the space provided.

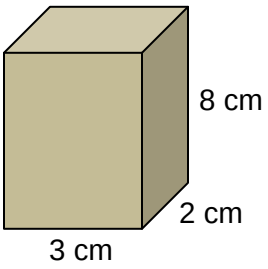
Volume of Prisms
Assessment 2

Find the volume of each figure shown. Show all your work in the space provided.

Cube with edges of 7 inches



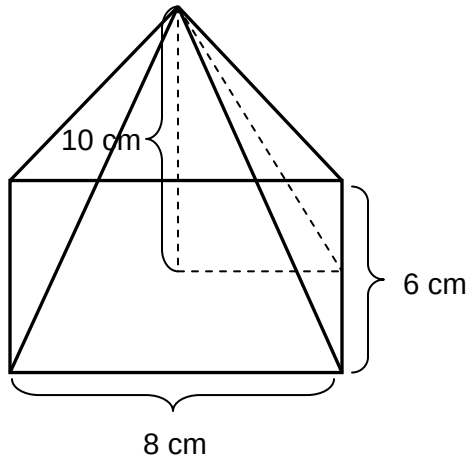
Rectangular prism with length of 3 cm, width of 2 cm, and height of 8 cm.



Volume of Pyramids
Assessment 3

Find the volume of the figure shown below. Show all your work in the space provided.

Rectangular pyramid with length of 8 cm, width of 6 cm, and height of 10 cm.



Extensions

1. Use modeling dough to construct three-dimensional objects, diagram them, and calculate their surface area and volume (Enrichment 1). Compare and contrast prisms and pyramids using modeling dough (Enrichment 2).

2. Put together three-dimensional objects of more than one shape and find the surface area and volume of the new object.

3. Find the surface area or volume of things you see everyday such as swimming pools, cans of food, boxes of cereal, etc. Calculate how much wrapping paper is necessary to wrap a present for a friend, given a particular size box.

Surface Area and Volume with Modeling Dough
Enrichment 1

1. Make a rectangular prism out of modeling dough. Make the prism large enough to measure and diagram. When you have completed making the figure, follow the steps below. Several worksheets are provided to help you diagram your figure and calculate the surface area and volume.
    a. Diagram the figure using a ruler and marking the measurements on a clean sheet of paper.
    b. Calculate the lateral surface area, total surface area, and volume of your figure. Be sure to show all your work.
    c. Indicate your answer for each on a sheet of paper by circling the answers or highlighting them.
2. Now make three of the following figures from your modeling dough. You must build at least one prism and at least one pyramid. Repeat the steps listed in #1a-1c above with each of these figures.
    a. cube
    b. triangular prism
    c. pentagonal prism
    d. hexagonal prism
    e. cone
    f. cylinder
    g. triangular pyramid
    h. square pyramid
    i. rectangular pyramid
    j. pentagonal pyramid
    k. hexagonal pyramid
3. Enrichment – Repeat steps 1a-1c using a sphere. Note that you cannot calculate lateral area of a sphere (only total surface area and volume).

$$\text{Formula for the Volume of a Sphere: } V = \frac{4}{3}\pi r^3$$

$$\text{Formula for the Total Surface Area of a Sphere: } T = 4\pi r^2$$

MODELING DOUGH WORKSHEETS

Diagram your rectangular prism and calculate the surface area in the space.

DIAGRAM	LATERAL SURFACE AREA
TOTAL SURFACE AREA	**VOLUME**

Diagram your figure and calculate the surface area in the space.

The figure I choose is a _____.

DIAGRAM	LATERAL SURFACE AREA
TOTAL SURFACE AREA	**VOLUME**

Diagram your figure and calculate the surface area in the space.

The figure I choose is a _____.

DIAGRAM	LATERAL SURFACE AREA
TOTAL SURFACE AREA	**VOLUME**

Diagram your figure and calculate the surface area in the space.

The figure I choose is a _____.

DIAGRAM	LATERAL SURFACE AREA
TOTAL SURFACE AREA	**VOLUME**

Surface Area and Volume
Comparing and Contrasting Prisms and Pyramids
Enrichment 2

Using modeling dough, construct a rectangular prism and a rectangular pyramid according to the given instructions. Then follow the instructions to repeat the same activity, constructing a cylinder and a cone.

RECTANGULAR PRISM AND RECTANGULAR PYRAMID

1. What is the difference between a prism and a pyramid?

2. Define the following two terms:

    a. rectangular prism

    b. rectangular pyramid

3. Using modeling dough, build a **rectangular prism**. Make certain that the prism is not too small to measure. In the space below, diagram the rectangular prism. Measure its height, width, and length and mark these measurements on your diagram. Make certain to indicate if the measurements are in centimeters or inches.

4. Calculate the lateral surface area of the prism showing all your work in the space provided.

5. Calculate the total surface area of the prism showing all your work in the space provided.

6. Calculate the volume of your prism showing all your work in the space provided.

7. Using modeling dough, build a **rectangular pyramid** that has exactly the same base and height as the rectangular prism you just built, diagrammed and measured. Diagram this pyramid in the space provided. Measure its height, width, and length and mark these measurements on your diagram. Make certain to indicate if the measurements are in centimeters or inches.

8. Calculate the lateral surface area of the pyramid showing all your work in the space provided.

9. Calculate the total surface area of the pyramid showing all your work in the space provided.

10. Calculate the volume of your prism showing all your work in the space provided.

11. Compare and contrast your findings for the rectangular prism with that for the rectangular pyramid. What is similar? What is different? Be very specific.

SIMILARITIES
DIFFERENCES

CYLINDERS AND CONES

1. What is the difference between a cylinder and a cone?

2. Define the following two terms:

   a. Cylinder

   b. Cone

3. Using modeling dough, build a **cylinder**. Make certain that the cylinder is not too small to measure. In the space below, diagram the cylinder. Measure its height and the radius or diameter of its base and mark these measurements on your diagram. Make certain to indicate if the measurements are in centimeters or inches.

4. Calculate the lateral surface area of the cylinder showing all your work in the space provided. Hint - Remember that a cylinder has two bases and that the area around the cylinder would be a rectangle if you could cut it out (think about the label on a can of soup.)

5. Calculate the total surface area of the cylinder showing all your work in the space provided.

6. Calculate the volume of your cylinder showing all your work in the space provided.

7. Using modeling dough, build a **cone** that has exactly the same base and height as the cylinder you just built, diagrammed and measured. Diagram this cone in the space provided. Measure its height and the radius or diameter of its base and mark these measurements on your diagram. Make certain to indicate if the measurements are in centimeters or inches.

Do you see notice anything interesting when looking at your work with the rectangular prism and rectangular pyramid and your work with the cylinder and cone? Explain your answer in the space provided.

Sources

2008 AZ Mathematics Standards
2006 NCTM Curriculum Focal Points
2000 NCTM Principles and Standards
2008 The Final Report of the National Mathematics Advisory Panel
1999 Bringing the NCTM Standards to Life, Exemplary Practices from High Schools
http://www.repertorium.net/rostau/measures.html

RECIPES FOR MODELING DOUGH

Traditional Dough

- 1 cup flour
- 1 cup warm water
- 2 teaspoons cream of tartar
- 1 teaspoon oil
- 1/4 cup salt
- food coloring

Mix all ingredients, adding food coloring last. Stir over medium heat until smooth. Remove from pan and knead until blended smooth. Place in plastic bag or airtight container when cooled. Properly stored, it will last for a long time.

No-Cook Dough

- 3 cups flour
- 3 cups salt
- 3 tablespoons alum

Combine ingredients and slowly add water, a little at a time. Mix well with spoon. As mixture thickens, continue mixing with your hands until it has the feel of clay. If it feels too dry, add more water. If it is too sticky, add equal parts of flour and salt. Store in an air-tight container or plastic bag.

Perfect Dough

- 2 cups flour
- 1 cups salt
- 2 cups water
- 2 tablespoons baby oil (works better than regular oil)
- 2 tablespoons cream of tartar
- liquid food coloring

Combine dry ingredients. Add water and oil. Stir well. Microwave on high 4 to 5 minutes. Stir again. Microwave another minute. Stir. Continue to microwave one minute, then stir, until dough is the consistency of mashed potatoes. Cool it enough to touch. Knead in food coloring until dough is desired color. Store in an air-tight container or plastic bag.

Polygons
An ADE Mathematics Lesson
Days 11-15

Author	ADE Content Specialists
Grade Level	10th grade
Duration	Five days

Aligns To	Connects To
Mathematics HS: **Strand 3: Patterns, Algebra, and Functions** **Concept 1: Patterns** **PO 1.** Recognize, describe, and analyze sequences using tables, graphs, words, or symbols; use sequences in modeling. **Concept 3: Algebraic Representations** **PO 1.** Create and explain the need for equivalent forms of an equation or expression. **Strand 4: Geometry and Measurement** **Concept 1: Geometric Shapes** **PO 2.** Visualize solids and surfaces in 3-dimensional space when given 2-dimensional representations and create 2-dimensional representations for the surfaces of 3-dimensional objects. **PO 3.** Create and analyze inductive and deductive arguments concerning geometric ideas and relationships. **PO 4.** Apply properties, theorems, and constructions about parallel lines, perpendicular lines, and angles to prove theorems. **PO 6.** Solve problems using angle and side length relationships and attributes of polygons. **PO 7.** Use the hierarchy of quadrilaterals in deductive reasoning.	**Mathematics HS:** **Strand 1: Number and Operations** **Concept 3: Estimation** **PO 2.** Use estimation to determine the reasonableness of a solution. **Strand 3: Patterns, Algebra, and Functions** **Concept 3: Algebraic Representations** **PO 2.** Solve formulas for specified variables. **Strand 4: Geometry and Measurement** **Concept 3: Coordinate Geometry** **PO 4.** Verify characteristics of a given geometric figure using coordinate formulas for distance, midpoint, and slope to confirm parallelism, perpendicularity, and congruence. **Strand 5: Structure and Logic** **Concept 2: Logic, Reasoning, Problem Solving, and Proof** **PO 3.** Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.

Aligns To	Connects To
Strand 5: Structure and Logic **Concept 2: Logic, Reasoning, Problem Solving, and Proof** **PO 1.** Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made. **PO 2.** Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s). **PO 4.** Generalize a solution strategy for a single problem to a class of related problems; explain the role of generalizations in inductive and deductive reasoning. **PO 5.** Summarize and communicate mathematical ideas using formal and informal reasoning. **PO 7.** Find structural similarities within different algebraic expressions and geometric figures.	

Overview

In this lesson you will recognize various forms of polygons and polyhedra, specifically working with comparing and contrasting the properties of regular polygons. You will also compare the properties of various types of quadrilaterals.

Purpose

You will learn the similarities and differences between regular polygons as well as between different forms of quadrilaterals. This will help you to work with irregular polygons and different forms of polygons and polyhedra.

Materials

- Regular polygon worksheets
- Quadrilateral worksheets
- Lined paper such as graph paper
- Ruler
- Compass and protractor

Objectives

Students will:

- Create and analyze inductive and deductive arguments concerning geometric ideas and relationships.
- Solve problems using angle and side length relationships and attributes of polygons.
- Use the hierarchy of quadrilaterals in deductive reasoning.

Lesson Components

Prerequisite Skills: In earlier grades, you learned to recognize various geometric figures and to assign attributes to these simple geometric figures.

Vocabulary: *polygon, polyhedra, polyhedron, regular polygon, quadrilateral, equiangular polygon, interior angle of a polygon, exterior angle of a polygon, diagonal of a polygon, quadrilateral, trapezoid, rhombus, isosceles trapezoid, rectangle, square, parallelogram*

Session 1 (2 days)

1. Compare and contrast polygons with polyhedra. Students compare and contrast various forms of regular polygons.

Session 2 (3 days)

1. Compare and contrast various forms of quadrilaterals.

Assessment

There are two assessments that will help you pinpoint misconceptions before moving on to more complex comparisons. The first assessment follows Session 1 and assesses knowledge of polygons. The second assessment follows Session 2 and assesses knowledge of quadrilaterals.

Polygons
Session 1 – Regular Polygons

Discover some fascinating characteristics of polygons.

1. Polygons have special names given in relationship to the number of sides the polygon has. List all the polygons you can and include how many sides each has.

2. What is the difference between a **polygon** and a **polyhedron**? Can you name some polyhedra and describe them? Draw one example of a polygon and one example of a polyhedron?

3. What is a **regular polygon**? Have you learned to construct any of them? If so, which ones?

4. Do you think that you can construct any of the other regular polygons? Try it and see what happens when you vary the number of sides of the polygon. Use your compass and ruler and just try to use some of the steps you already know.

5. What characteristics would an **equiangular polygon** exhibit? Is an equiangular polygon the same thing as a regular polygon? Support your answer.

6. Draw three different regular polygons. Either construct the polygon to ensure uniformity or use a protractor and ruler to ensure uniformity. You will record the information in chart form in question 7.
    a. Measure each interior angle. Label each angle using a lowercase letter.
    b. Extend the sides to form the exterior angles. Measure each exterior angle. Label each angle using a number.
    c. Draw all diagonals, label and count them.

Polygon 1

Polygon 2

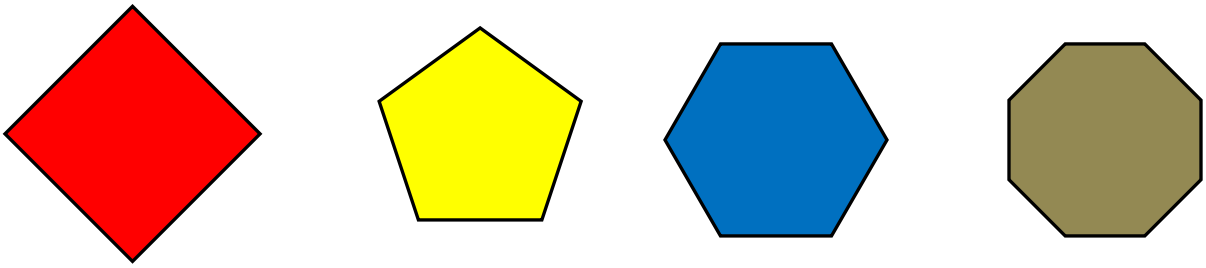
Polygon 3

7. Now consolidate all the information you obtained in question 6 in the chart provided.

Name of Polygon	Number of sides	Total Interior angle measure	Total Exterior angle measure	Number of Diagonals

8. Draw two additional regular polygons and repeat all steps in number 6 above. Add this information to your chart.

9. Experiment with all the information you obtained to see what new theorems or relationships emerge. Can you recognize three new concepts from the chart previously unknown to you? You must support your claims with visible evidence that you obtained while working on the regular polygons.



Hints

Consider relationships, for example the number of sides to number of diagonals.

Look for patterns.

Look only at the interior angles, then the exterior angles, then the diagonals. Then, look at combinations of these things.

Be creative. Allow flexibility. Be as accurate as possible with the measurements.

Arrange the entries in your chart in some type of order.

Use colored pencils and other ways of distinguishing angles, lines, diagonals.

Make sure your work is neat and that you understand what your chart is telling you.

Polygons
Assessment 1 – Regular Polygons

1. Complete the following questions. Assume all polygons are regular polygons.

   The total measure of all the interior angles of a triangle is _____ degrees.

   A parallelogram has _____ sides.

   A rhombus has _____ diagonals.

   The sum of the measure of the exterior angle of a trapezoid is _____ degrees.

   The total measure of all the interior angles of a pentagon is _____ degrees.

   A hexagon has _____ diagonals.

   The total measure of the exterior angles of an octagon is _____ degrees.

2. State the relationship between the number of sides and number of diagonals for regular polygons.

3. Is there a relationship between the total measure of the interior angles and exterior angles of a regular polygon? Is this supported by the patterns that can be seen in examining quadrilaterals (parallelogram, rhombus, and trapezoid)? Why or why not?

4. Define the difference between a polygon and a polyhedron. Draw an example of 2 different polyhedra and label each of them with their name.

Polygons
Session 2 - Quadrilaterals

1. Define each of the following terms in the space provided:

Term	Definition
Quadrilateral	
Polygon	
Vertices of a polygon	
Sides of a polygon	
Diagonals of a polygon	
Isosceles trapezoid	
Leg of a trapezoid	
Base of a trapezoid	
Base angle	

2. Define and draw examples of each of the following using a protractor and ruler or by using a construction in the space provided.
    a. rectangle
    b. square
    c. trapezoid
    d. rhombus
    e. parallelogram

3. Make a chart in which you compare the properties of all of the figures in 2a-2e. Include the number and types of angles and sides. What is the relationship of various sides of one quadrilateral? What do the figures have in common? What are the differences between the figures?

Quadrilateral	Types of Sides	Types of Angles	Similarities	Differences
Rectangle				
Square				
Trapezoid				
Rhombus				
Parallelogram				

4. Consider the following statements and determine if they are true. Justify your answer. If the answer is false, modify it to make a true statement.

   a. All rhombuses are squares.

   b. All trapezoids have one set of parallel sides.

   c. All square are rectangles.

   d. All rectangles are parallelograms.

   e. The diagonals of a parallelogram are always perpendicular to each other.

f. The sides of an isosceles trapezoid are always congruent.

g. The only type of quadrilateral that does not have opposite sides parallel is a trapezoid.

Polygons
Assessment 2 – Quadrilaterals

1. Match each term to its definition.

_____ Quadrilateral

_____ Polygon

_____ Vertices of a polygon

_____ Sides of a polygon

_____ Diagonals of a polygon

_____ Isosceles trapezoid

_____ Leg of a trapezoid

_____ Base of a trapezoid

_____ Base angle

A. the points at which two sides of a polygon meet

B. either of the two sides which connect the parallel bases of a trapezoid

C. the angle that is formed by the base and one leg in a polygon

D. a polygon with four sides

E. the segment joining two adjacent vertices in a figure

F. a closed two-dimensional figure made up of segments which intersect only at the segment endpoints

G. a side of a trapezoid that is perpendicular to its height

H. a trapezoid in which the non-parallel sides are congruent

I. a line segment joining two non-adjacent vertices of a polygon

2. Complete the following chart comparing the similarities and differences between the two types of quadrilaterals named. Justify your answers.

QUADRILATERALS	SIMILARITIES	DIFFERENCES
Square and Parallelogram		
Rhombus and Rectangle		
Trapezoid and Parallelogram		

3. Draw a graphic organizer or some other pictorial representation that shows the relationship between the various types of quadrilaterals. Make certain to include a parallelogram, trapezoid, rhombus, square, and rectangle.

4. Consider the following statements and determine if they are true or false. If the answer is false, modify it to make a true statement.

    a. All squares are rhombuses.

    b. All trapezoids have one set of parallel sides.

    c. All rectangles are squares.

    d. All rectangles are parallelograms.

    e. The legs of an isosceles trapezoid are not congruent.

Extensions

1. Use applets to further explore the relationships between interior and exterior angles of polygons.

   Triangles
   http://www.geogebra.org/en/upload/files/english/Barbara_Perez/Triangle_Exterior.html

   Quadrilaterals
   http://www.geogebra.org/en/upload/files/english/Barbara_Perez/Quadrilateral_Exterior.html

   Pentagons
   http://www.geogebra.org/en/upload/files/english/Barbara_Perez/Pentagon_Exterior.html

   Hexagons
   http://www.geogebra.org/en/upload/files/english/Barbara_Perez/Hexagon_Exterior.html

2. Work with irregular polygons to determine similarities and differences between different forms of irregular polygons.

Sources

1999 Bringing the NCTM Standards to Life, Exemplary Practices from High Schools, p. 10-14
2000 NCTM Principles and Standards, p. 308-323
2008 AZ Department of Mathematics Standards
2008 The Final Report of the National Mathematics Advisory Panel, p. 20, 29-30

Geometric Probability
An ADE Mathematics Lesson
Days 16-20

Author	ADE Content Specialists
Grade Level	10th grade
Duration	Five days

Aligns To

Mathematics HS:
Strand 1: Number and Operations
Concept 2: Numerical Operations
PO 3. Calculate powers and roots of rational and irrational numbers.
Concept 3: Estimation
PO 1. Determine rational approximations of irrational numbers.

Strand 2: Data Analysis, Probability, and Discrete Mathematics
Concept 2: Probability
PO 1. Make predictions and solve problems based on theoretical probability models.
PO 5. Use concepts and formulas of area to calculate geometric probabilities.

Strand 4: Geometry and Measurement
Concept 1: Geometric Shapes
PO 1. Use the basic properties of a circle (relationships between angles, radii, intercepted arcs, chords, tangents, and secants) to prove basic theorems and solve problems.
Concept 4: Measurement
PO 2. Find the length of a circular arc; find the area of a sector of a circle.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.
PO 2. Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s).
PO 4. Generalize a solution strategy for a single problem to a class of related problems; explain the role of generalizations in inductive and deductive reasoning.
PO 5. Summarize and communicate mathematical ideas using formal and informal reasoning.

Connects To

Mathematics HS:
Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 2. Solve formulas for specified variables.

Strand 4: Geometry and Measurement
Concept 4: Measurement
PO 3. Determine the effect that changing dimensions has on the perimeter, area, or volume of a figure.
PO 4. Solve problems involving similar figures using ratios and proportions.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 3. Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.

Overview

It is sometimes necessary to find the probability of an event that involves geometric figures. This lesson will demonstrate how to find geometric probability given various shapes and figures.

Purpose

The purpose of this lesson is to give you more experience working with probability in different problems.

Materials

- Geometric Probability Worksheets
- Ruler

Objectives

Students will:

- Make predictions and solve problems based on theoretical probability models.
- Use concepts and formulas of area to calculate geometric probabilities.

Lesson Components

Prerequisite Skills: In 7th and 8th grades you learned to calculate areas of various figures including composite figures. You learned how to calculate probability given dependent and independent events.

Vocabulary: *probability, geometric probability, dependent event, independent event, expected outcome, composite figure, area*

Session 1 (2 days)

1. Review simple probability.

Session 2 (3 days)

1. Use area formulas and various concepts to calculate geometric probabilities.

Assessment

There is one assessment that follows Session 2 and assesses knowledge of geometric probabilities.

Geometric Probability
Session 1 - Review of Probability

Before working with geometric probability concepts, let's do a short review of some basic probability. It is necessary to understand the terms that follow to be able to work with geometric probability. If you are having difficulty understanding any of them, please review basic probability. Some examples follow the definitions.

- **Theoretical probability** is the likelihood an event will occur under ideal circumstances divided by the total possible outcomes.
- An **outcome** is a possible result for a probability experiment or simulation.
- A **sample space** is a list of all possible outcomes of an activity.
- **Experimental (empirical) probability** is a ratio formed by the comparison of the number of times an event occurs in an experiment to the number of times the experiment is completed.
- **Complementary events** are two events whose probabilities of occurring sum to one; mutually exclusive events (e.g., when flipping a coin, getting a head and getting a tail are complementary events).
- **The Law of Large Numbers** states that the larger the sample the closer the experimental probability will approximate the theoretical probability.
- **Independent events** are two events in which the outcome of the second event is not affected by the outcome of the first event.
- **Dependent events** are two events such that the likelihood of the outcome of the second event is affected by the outcome of the first event.

Theoretical probability is generally calculated before an experiment is conducted. Once the experiment has been completed, the experimental probability can be determined. That experimental probability will not alter the theoretical probability. The larger the sample is, the more likely that the experimental probability will approach the theoretical probability which is called the Law of Large Numbers.

Example 1:

What is the probability that a wooden cube, numbered 1 to 6 on each of its faces, will be a 6 when a person tosses that cube?

Solution:

List the possible outcomes when a wooden cube containing the numbers one to 6 is tossed. There are only 6, which are the following:

{1, 2, 3, 4, 5, 6}

Theoretical probability tells us that the probability of an event is the ratio of the possible number of correct outcomes to the total number of outcomes. In this case, there is one correct outcome (6) over six (1, 2, 3, 4, 5, 6) possible outcomes.

Our answer is 1:6 or can also be written as $\frac{1}{6}$.

Example 2:

What is the probability that the sum of two wooden cubes, each numbered 1 to 6 on their faces, will be a 7 when two cubes are tossed?

Solution:

List the possible outcomes when two wooden cubes containing the numbers 1 to 6 are tossed. There are only 36 outcomes, which are shown in the table; 6 of the outcomes show a sum of 7.

Sum	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Theoretical probability tells us that the probability of an event is the ratio of the possible number of correct outcomes to the total number of outcomes. In this case, there are six correct outcomes (6) over 36 possible outcomes.

Our answer is 6:36 or can also be written as $\frac{6}{36}$. We can also write this in reduced form. Then our answer becomes 1:6 or $\frac{1}{6}$. You may see probability answers written in reduced form sometimes and at other times not in reduced form.

Example 3:

Sometimes the second event in which we are trying to determine probability depends on what happens in the first event. A candy store has a jar on the counter that contains lollipops. The owner of the store places 10 red lollipops, 5 green lollipops, 5 yellow lollipops and 5 orange lollipops in the jar every morning. Every child can take one free lollipop before leaving the store.

- What is the theoretical probability that the first child will choose a green lollipop?
- Assuming the first child chose a green lollipop, what is the theoretical probability that the second child will choose a red lollipop?
- Assuming the first child chose a green lollipop and the second child chose a red lollipop, what is the theoretical probability that the third child that day will choose a lollipop that is not red?

Solution:

Let's consider the answer to the first question. There are

10 *(red)* + 5 *(green)* + 5 *(yellow)* + 5 *(orange)* = 25 total lollipops.

The first child will choose one of these 25 lollipops of which 5 are green.

The probability of the child picking a green lollipop is $\frac{5}{25}$ or $\frac{1}{5}$. We can also represent this by 5:25 or 1:5.

When the first child takes this one lollipop out of the jar, there are 24 lollipops left. Of these 24 lollipops, 10 are red. The theoretical probability of the second child taking a red lollipop out of the jar is $\frac{10}{24} = \frac{5}{12}$ which can also be represented by 10:24 or 5:12.

When the second child takes this lollipop, there are 23 lollipops left in the jar. Nine of these will be red which means that 14 of them will be other colors. The probability of the third child taking a lollipop that is not red is $\frac{14}{23}$ which can be represented by 14:23.

Now you try one problem.

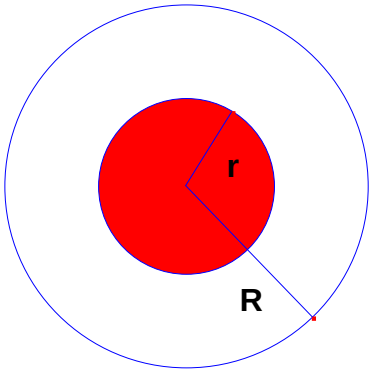
Four children are playing a game that involves a spinner containing three different colors. Half of the spinner is red. One fourth of the spinner is blue and one fourth of the spinner is yellow. What is the probability that one of the children will land on the yellow sector of the spinner? (Hint – draw the spinner to help you.) Show your work in the space provided.

Geometric Probability
Session 2

Geometric probability is the likelihood of an event occurring based on geometric relationships such as area, surface area, or volume.

Example 1:

If an arrow hits the target, what is the probability of hitting the red (shaded) bull's eye?



Solution:

The area of the white circle with radius R is πR^2.

The area of the red (shaded) bulls eye with radius r is πr^2.

Theoretical probability is the likelihood an event will occur under ideal circumstances divided by the total possible outcomes.

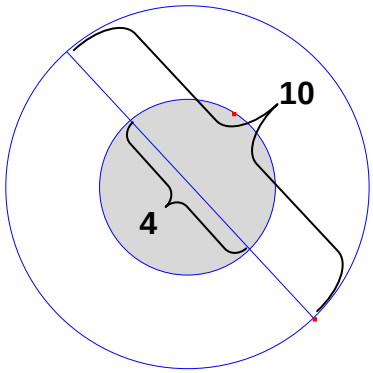
If A represents area, then A_{red} represents the area of the red circle and A_{white} represents the area of the white circle. The probability of hitting the red (shaded) bull's eye is

$$\frac{A_{red}}{A_{white}}$$

If an arrow hits the target, the probability of hitting the red (shaded) bull's eye is $\frac{\pi r^2}{\pi R^2} = \frac{r^2}{R^2}$

Example 2:

If an arrow hits the target, what is the probability of hitting the gray (shaded) bull's eye if the diameter of the white circle is 10 and the diameter of the gray circle is 4?



Solution:

The area of the white circle with diameter of 10 is $\pi\left(\frac{10}{2}\right)^2 = 25\pi$.

The area of the gray (shaded) bulls eye with diameter 4 is $\pi\left(\frac{4}{2}\right)^2 = 4\pi$.

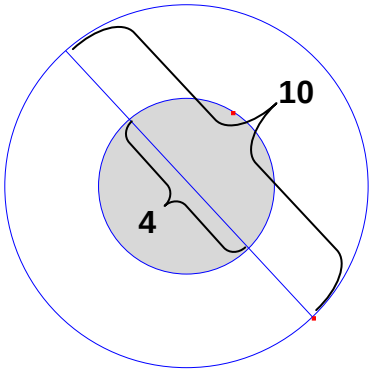
If A represents area, then A_{red} represents the area of the gray circle and A_{white} represents the area of the white circle. The probability of hitting the gray (shaded) bull's eye is

$$\frac{A_{red}}{A_{white}} = \frac{4\pi}{25\pi} = \frac{4}{25}$$

If an arrow hits the target, the probability of hitting the gray (shaded) bull's eye is $\frac{4}{25}$.

Example 3:

If an arrow hits the target, what is the probability of hitting the white part of the target if the diameter of the white circle is 10 and the diameter of the gray circle is 4?



Solution:

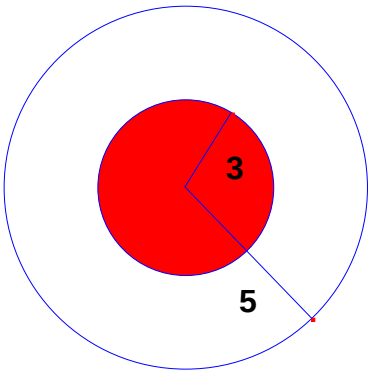
The area of the white circle minus the area of the gray circle is $25\pi - 4\pi = 21\pi$.

The area of the white circle is 25π.

The probability of hitting the white part of the target is $\frac{21\pi}{25\pi} = \frac{21}{25}$.

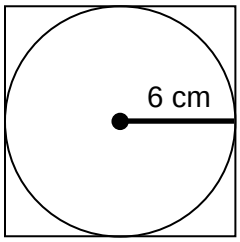
Now you try one:

Problem 1: If an arrow hits the target, what is the probability of hitting the white part of the target if the radius of the white circle is 5 and the radius of the red circle is 3? Show your work in the space provided.



Example 4:

If you choose a point in the square, what is the probability that it is not in the circle?



Solution:

The area of the square is represented by $A = s^2 \Rightarrow A = (2 \bullet 6)^2 = 12^2 = 144\, cm^2$.

The area of the circle is represented by $A = \pi r^2 \Rightarrow A = \pi 6^2 = 36\pi\, cm^2$.

Theoretical probability is the likelihood an event will occur under ideal circumstances divided by the total possible outcomes.

The area of the part of the figure that is not in the circle is $144 - 36\pi$.

The probability that a chosen point is not in the circle is $\frac{144 - 36\pi}{144} = \frac{36(4-\pi)}{36 \bullet 4} = \frac{4-\pi}{4}$.

What is the probability that a chosen point is in the circle?

Remember that the probability of complementary events must have a sum that totals 1.

- Therefore, the probability that the chosen point is in the circle is:

  $$1 - \frac{4-\pi}{4} = \frac{4}{4} - \frac{4-\pi}{4} = \frac{4-(4-\pi)}{4} = \frac{4-4+\pi}{4} = \frac{\pi}{4}.$$

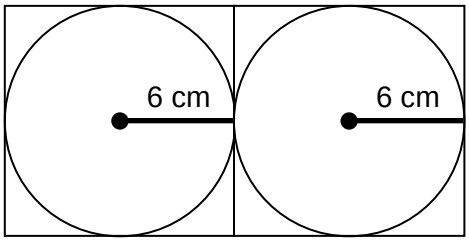
- A second way to calculate the probability is to divide the probability of finding a point in the circle by the probability of finding a point in the square. We can represent this by:

  $$\frac{36\pi}{144} = \frac{\pi}{4}.$$

We obtained the same answer in two different ways.

Example 5:

If you choose a point in the figure, what is the probability that the point is in one of the two circles?



Solution:

The area of the 2 squares is represented by

$A = 2 \bullet s^2 \Rightarrow A = 2 \bullet (2 \bullet 6)^2 = 2 \bullet 12^2 = 2 \bullet 144 = 288\,\text{cm}^2$.

The area of the 2 circles is represented by $A = 2 \bullet \pi r^2 \Rightarrow A = 2 \bullet \pi 6^2 = 2 \bullet 36\pi = 72\pi$.

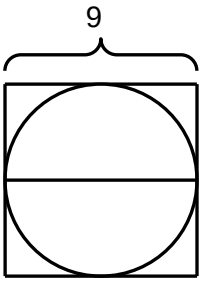
The probability that the point is in the circles is $\frac{72\pi}{288} = \frac{\pi}{4}$.

Note that this is the answer to the same problem with one square and one circle. Do you think this is correct? Justify your answer.

Now you try some.

Problem 2:

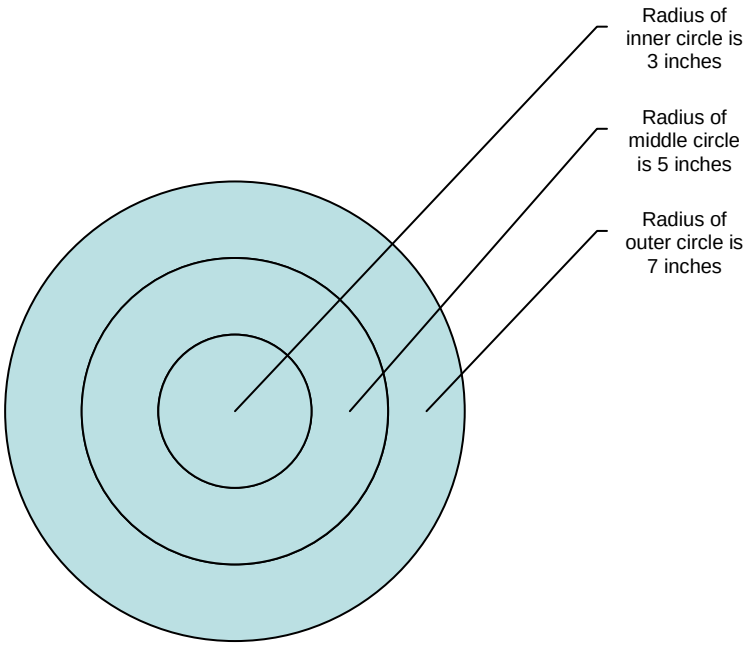
If you choose a point in the square, what is the probability that it is not in the circle? Show your work in the space provided.



Solution:

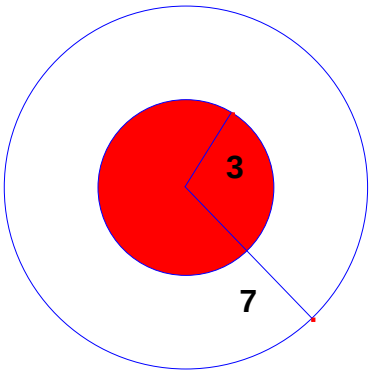
Problem 3:

What is the probability that a dart thrown at the target will land in the middle circle?

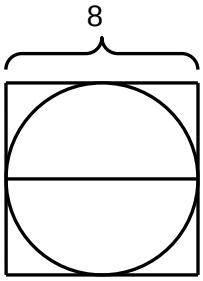


Geometric Probability
Assessment 1

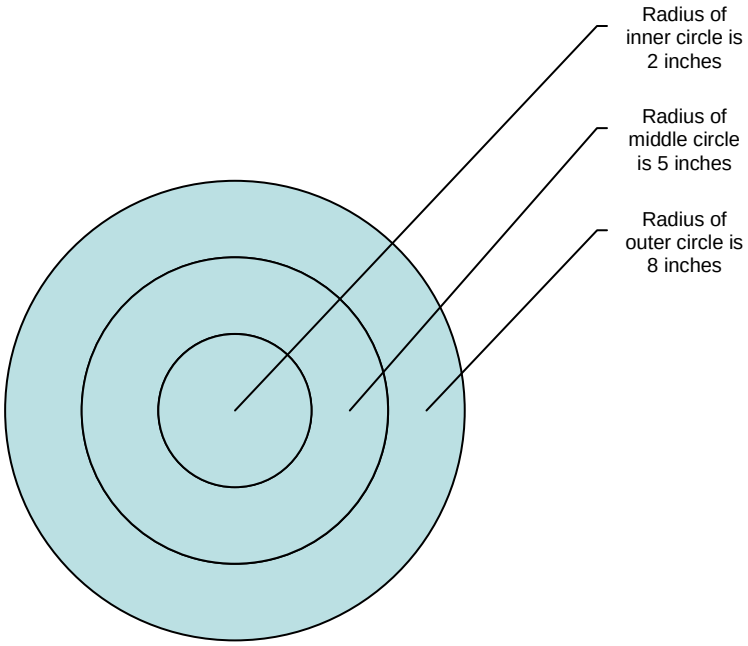
1. If an arrow hits the target, what is the probability of hitting the white part of the target if the radius of the white circle is 7 and the radius of the red circle is 3? Show your work in the space provided.



2. If you choose a point in the square, what is the probability that it is not in the circle? Show your work in the space provided.



3. What is the probability that a dart thrown at the target will land in the outer circle?

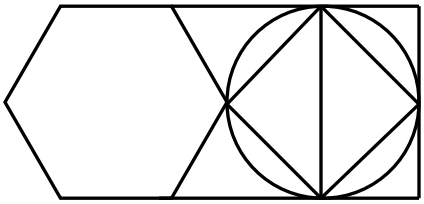


Extensions

1. Determine the probability given a more complicated composite figure. An enrichment activity is included in this lesson.

Geometric Probability
Enrichment

Consider the following composite figure and answer the questions that follow.



1. Describe the shapes that make up the composite figure.

2. Describe how to find the area of every shape that makes up the composite figure that you described in your answer to question 1.

3. Describe the process you would use to find the probability of a choosing a point at random that was not in the circle in the composite figure.

Sources

2008 AZ Mathematics Standards
2000 NCTM Principles and Standards, p. 331-333
2008 The Final Report of the National Mathematics Advisory Panel, p. 17
1999 Bringing the NCTM Standards to Life, Exemplary Practices from High Schools, p. 10-17

Congruence and Similarity of Triangles
An ADE Mathematics Lesson
Days 21-30

Author	ADE Content Specialists
Grade Level	10th grade
Duration	Ten days

Aligns To

Mathematics HS:
Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representation
PO 1. Create and explain the need for equivalent forms of an equation or expression.

Strand 4: Geometry and Measurement
Concept 1: Geometric Shapes
PO 3. Create and analyze inductive and deductive arguments concerning geometric ideas and relationships.
PO 4. Apply properties, theorems, and constructions about parallel lines, perpendicular lines, and angles to prove theorems.
PO 6. Solve problems using angle and side length relationships and attributes of polygons.
PO 8. Prove similarity and congruence of triangles.
PO 10. Solve problems using right triangles, including special triangles.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.
PO 2. Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s).
PO 4. Generalize a solution strategy for a single problem to a class of related problems; explain the role of generalizations in inductive and deductive reasoning.
PO 5. Summarize and communicate mathematical ideas using formal and informal reasoning.
PO 7. Find structural similarities within different algebraic expressions and geometric figures.

Connects To

Mathematics HS:
Strand 1: Number and Operations
Concept 3: Estimation
PO 2. Use estimation to determine the reasonableness of a solution.

Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 2. Solve formulas for specified variables.

Strand 4: Geometry and Measurement
Concept 2: Transformation of Shapes
PO 3. Sketch and describe the properties of a 2-dimensional figure that is the result of two or more transformations.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 3. Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.

Overview

In this lesson, you will learn how to determine whether two triangles are similar or congruent and will be able justify their answer.

Purpose

It is often necessary to determine the relationship between two geometric figures to solve a problem. After this lesson, you will be able to determine the relationship between two given triangles to determine if they are similar, congruent, or neither to assist with further problem-solving in geometry.

Materials

- Similarity and Congruence worksheets
- Protractor
- Scissors
- Ruler

Objectives

Students will:

- Solve problems using angle and side length relationships and attributes of polygons.
- Prove similarity and congruence of triangles.
- Determine different methods to prove similarity and congruence of triangles.

Lesson Components

Prerequisite Skills: In 6th through 9th grades, you learned to recognize various attributes associated with triangles. You made determinations as to the nature of these triangles (right, isosceles, equilateral). You learned how to measure the sides and angles of various triangles and learned the Triangle-Sum Theorem and the Triangle-Inequality Theorem.

Vocabulary: *triangle, right triangle, isosceles triangle, equilateral triangle, scalene triangle, side of a triangle, angle of a triangle, vertex, hypotenuse, leg, similarity, congruence*

Session 1 (5 days)

1. Determine if two triangles are congruent by several different methods including physically measuring the triangles and using formulas for determining the congruence of triangles such as SSS, SAS, ASA, AAS, and H-L.

Session 2 (5 days)

1. Determine if two triangles are similar by several different methods including physically measuring the triangles and using formulas for determining the similarity of triangles such as SSS, SAS, and AA.

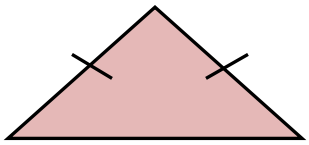
Assessment

There are two assessments that will help pinpoint misconceptions before moving on to more complex comparisons. The first assessment is focused on congruence of triangles and the second assessment is focused on similarity of triangles.

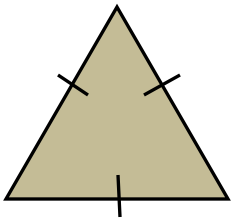
Congruence and Similarity of Triangles
Session 1 – Congruence

Review - Types of Triangles

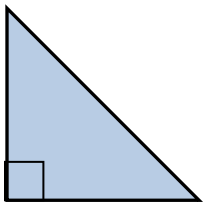
An **isosceles triangle** is a triangle that has two or more congruent sides (note: equilateral triangles are a subset of isosceles triangles).



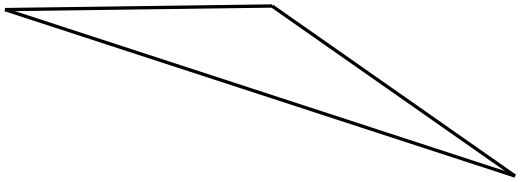
An **equilateral triangle** is a triangle in which all sides are congruent.



A **right triangle** is a triangle that contains a right angle.



A **scalene triangle** is a triangle with no congruent sides.



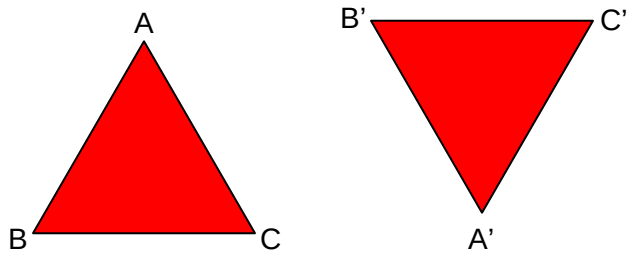
We can find the relationship of one triangle to another by making determinations of the size and shape of each triangle. Two triangles that have the same shape are said to be similar triangles. Two triangles that have the same size and shape are said to be congruent triangles. All congruent triangles are also similar triangles but not all similar triangles are congruent triangles.

Congruent triangles have the same shape and exactly the same size. We use the symbol $\cong$ to represent "is congruent to".

Similar triangles are triangles that have the same shape and are related in size by a scale factor. We use the symbol **~** to represent "is similar to".

Study the following examples of congruent triangles.

Example 1:



If $\Delta\ ABC \cong \Delta\ A'B'C'$, then the following relationships hold:

- $AB \cong A'B'$, $AC \cong A'C'$, $BC \cong B'C'$
- $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$
- $\angle BAC \cong \angle B'A'C'$, $\angle ABC \cong \angle A'B'C'$, $\angle ACB \cong \angle A'C'B'$

Example 2:

If two triangles are congruent and one triangle is an equilateral triangle, the other triangle must also be an equilateral triangle. Why would this be a true statement?

If two triangles are congruent and one triangle is an isosceles triangle, the other triangle must also be an isosceles triangle. Why would this be a true statement?

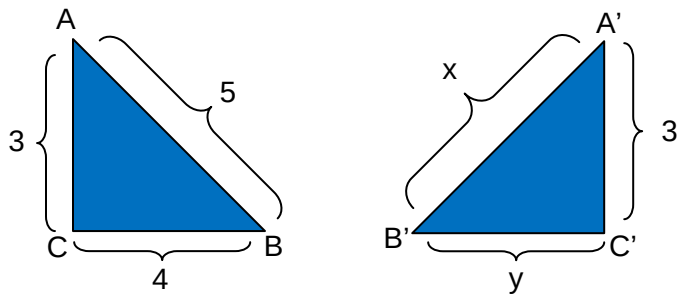
If two triangles are congruent and one triangle is a right triangle, the other triangle must also be a right triangle. Why would this be a true statement?

If two triangles are congruent and one triangle is a scalene triangle, the other triangle must also be a scalene triangle. Why would this be a true statement?

Can you summarize all the statements in the four parts of Example 2 into one or two succinct statements?

Example 3:

Fill in the measures of the missing sides, x and y, of the right triangle given that $\Delta\ ABC \cong \Delta\ A'B'C'$.

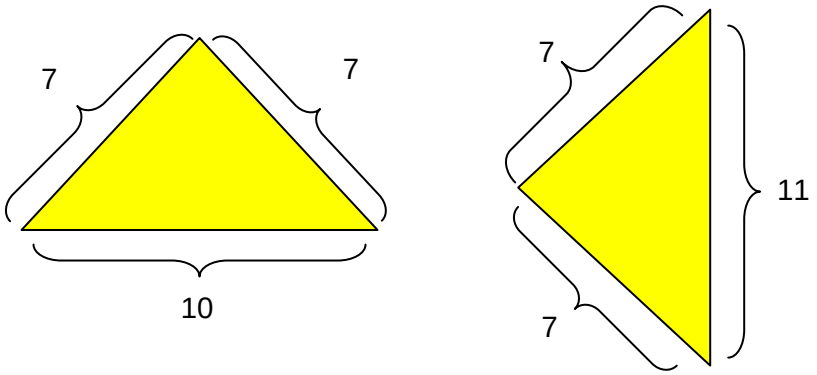


Example 4:

The measures of two angles of a triangle are 30° and 45°. The measures of a second triangle are also 30° and 45°. Are the two triangles congruent? Justify your answer.

Example 5:

Determine if the following two triangles are congruent. Justify your answer.



We can determine if two triangles are congruent by several different postulates.

Two triangles are congruent if each side of one triangle is congruent to each side of the other triangle. **(SSS)**

Two triangles are congruent if two sides of one triangle and the included angle of one triangle are congruent to two sides and the included angle of the other triangle. **(SAS)**

Two triangles are congruent if two angles of one triangle and the included side of one triangle are congruent to two angles of one triangle and the included side of the other triangle. **(ASA)**

Two triangles are congruent it two angles of one triangle and the non-included side are congruent to two angles of another triangle and the non-included side. **(AAS)**

Two right triangles are congruent if the hypotenuse and leg of one triangle are congruent to the hypotenuse and leg of the other triangle. **(H-L)**

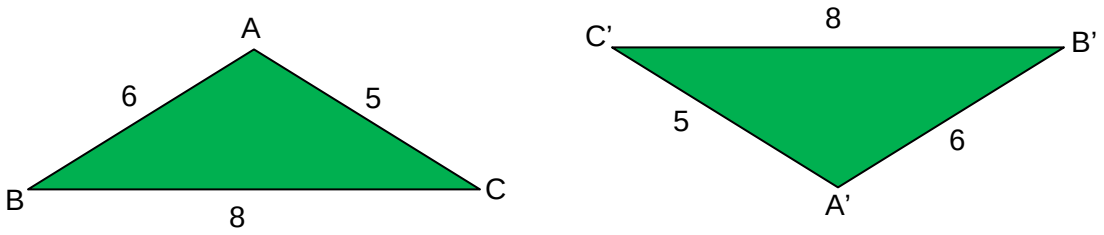
We can also make statements about the angles and sides of congruent triangles.

Corresponding **p**arts of **c**ongruent **t**riangles are **c**ongruent **(CPCTC)**

(When triangles are congruent, each corresponding side has the same measure and each corresponding angle has the same measure.)

Example 6:

Look at the following triangles and determine if they are congruent by one of the congruence postulates. Justify your answer. What statements can you make about the angles of the two triangles?

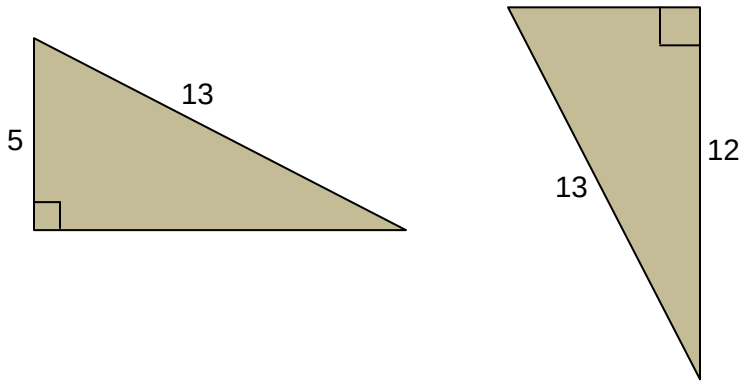


Example 7:

Are all 30°-60°-90° triangles congruent? Justify your answer.

Example 8:

Look at the following triangles and determine if they are congruent by one of the congruence postulates. The hypotenuse of each triangle is 13. The shortest leg in the first triangle is 5 and the longest leg of the second triangle is 12. Justify your answer. What statements can you make about the missing sides and angles of the two triangles?



Make up a congruence problem based on one of the congruence postulates. Show the problem in the space provided. Show at least one correct method to solve the problem you created.

Congruence and Similarity of Triangles
Assessment 1 – Congruence

Write out the meaning of each congruence postulate in the space provided.

Postulate	Meaning
SAS	
AAS	
SSS	
SAS	
H-L	
CPCTC	

Determine if the following sets of triangles are congruent. Justify your answer.

Problem 1:

The hypotenuse of each triangle is 5. The shortest leg in the first triangle is 3 and the longest leg of the second triangle is 4.

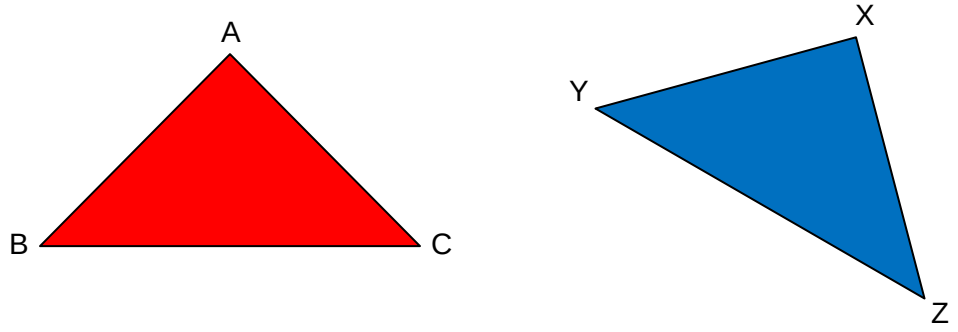
Problem 2:

The shortest sides of two triangles are 4 and 5, respectively. The angle in between the two sides of the first triangle is 48 degrees and the angle in between the two sides of the second triangle is 42 degrees. Which is a true statement about the relationship of the two triangles? Justify your answer.

1. The two triangles are congruent by SAS.
2. The two triangles can never be congruent because the angle in between the two given sides is different.
3. There is not enough information to determine if the two triangles are congruent.

Problem 3:

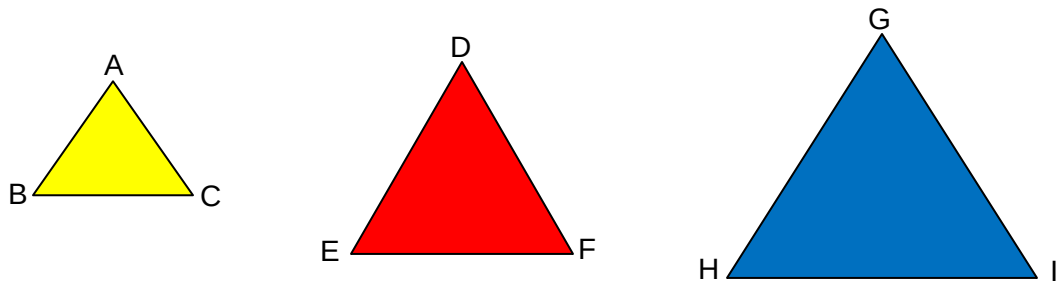
If $AB = 4$, $BC = 8$, $\angle C = 40°$ and $YZ = 8$, $XY = 4$ and $\angle Y = 40°$, is $\Delta ABC \cong \Delta XYZ$? Why or why not?



Congruence and Similarity of Triangles
Session 2 – Similarity

Sometimes, one triangle has exactly the same shape as another triangle but is not the same size. Consider the case of equilateral triangles. In equilateral triangles, all of the triangles have exactly the same shape, because all angles are 60°. All sides of an equilateral triangle are the same. Are all equilateral triangles then similar? Remember that similar triangles are triangles that have the same shape and are related in size by a scale factor. We use the symbol **~** to represent "is similar to".

Consider the following set of equilateral triangles.



Example 1:

In the yellow triangle, ΔABC, all sides are 2. In the red triangle, ΔDEF, all sides are 3. In the blue triangle, ΔGHI, all sides are 5. Are any of the triangles similar to the others in the set?

Solution:

Compare the yellow and red triangles. Each corresponding side of the yellow triangle, ΔABC, has a ratio of 2:3 to the corresponding side of the red triangle, ΔDEF. In other words, each side of ΔABC is $\frac{2}{3}$ of each side of ΔDEF.

Therefore, we know that the yellow triangle is similar to the red triangle and we write that $\Delta ABC \sim \Delta DEF$.

Compare the red and blue triangles. Each corresponding side of the red triangle, ΔDEF, has a ratio of 3:5 to the corresponding side of the blue triangle, ΔGHI. In other words, each side of ΔDEF is $\frac{3}{5}$ of each side of ΔGHI.

Therefore, we know that the red triangle is similar to the blue triangle and we write that $\Delta DEF \sim \Delta GHI$.

Is the yellow triangle similar to the blue triangle?

Following the same logic used to compare ΔABC with ΔDEF and ΔDEF with ΔGHI, we can compare ΔABC with ΔGHI.

The sides of ΔABC are each 2 units. The sides of ΔGHI are each 5 units. Therefore each corresponding side of ΔABC is $\frac{2}{5}$ of the corresponding side of ΔGHI. We know that the ratio of the corresponding sides of the yellow triangle to the corresponding sides of the blue triangle is 2:5.

Therefore, we know that the yellow triangle is similar to the blue triangle and we write that $\Delta ABC \sim \Delta GHI$.

The similarity postulate is known as **SSS** and states that if three sides of a triangle are in the same ratio to three corresponding sides of another triangle, the two triangles are similar.

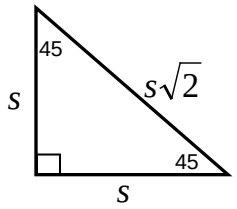
Example 2:

Are special right triangles similar to each other? Examine both 45°-45°-90° triangles and 30°-60°-90° triangles**.**

Solution:

Case 1: 45°-45°-90° triangles

What do we know about the special relationships in 45°-45°-90° triangles?



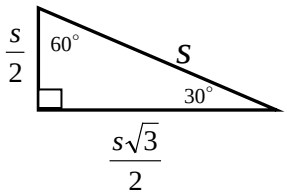
We know that the relationship of each leg of the right triangle to the hypotenuse is

$s : s\sqrt{2}$ or 1: $1\sqrt{2}$ This means that the hypotenuse of a right triangle is $\sqrt{2}$ times the leg. This ratio is constant no matter what the measure of the hypotenuse is. Therefore, we know that in 45°-45°-90° triangles, all angles are the same and that corresponding sides have the same ratio.

Therefore all 45°-45°-90° triangles are similar.

Case 2: 30°-60°-90° triangles

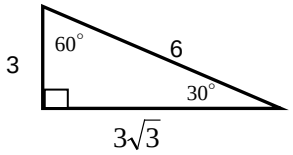
What do we know about the special relationships in 30°-60°-90° triangles?

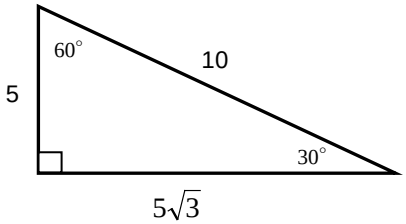


If we compare the side of one 30°-60°-90° triangle to the side of the corresponding side of another triangle, do we obtain the same ratio?

We will not spend time in this lesson proving the special relationships in these special right triangles. We will talk about the necessary conditions for the two 30°-60°-90° triangles to be similar and if they are met. If the hypotenuse of the triangle is *s*, then the side opposite the 30° angle will always be half of the measure of the hypotenuse. The side opposite the 60° angle will always be one half of the measure of the hypotenuse times $\sqrt{3}$.

Let's look at an example of two different 30°-60°-90° triangles.





The ratios of the corresponding sides of the two triangles are $\frac{3}{5} = \frac{3\sqrt{3}}{5\sqrt{3}} = \frac{6}{10}$.

These ratios all reduce to $\frac{3}{5}$. Since the ratios of the corresponding sides are the same, the two special right triangles are similar by SSS. However, this is just an example and does not prove that all 30°-60°-90° triangles are similar. We can follow the same process though and show that the ratios of the corresponding sides of these triangles will always be the same and therefore all 30°-60°-90° will be similar.

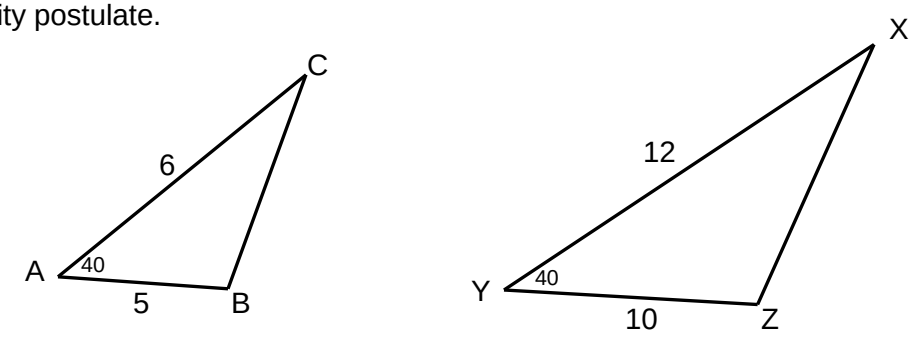
Show one example of two 45°-45°-90° triangles and one example of two 30°-60°-90° triangles that are similar. Justify your work.

45°-45°-90° triangle
30°-60°-90° triangle

The similarity postulate **SAS** states that if two sides and the included angle of one triangle are in the same ratio as the two corresponding sides and the included angle of the other triangle, the two triangles are similar.

Example 2:

Examine the two triangles below and determine if ΔABC is similar to ΔXYZ by the SAS similarity postulate.



The SAS triangle states that the ratio of two corresponding sides must be in the same ratio and included angle between the corresponding sides must be the same.

Note that $\frac{6}{12} = \frac{5}{10}$ since both reduce to $\frac{1}{2}$.

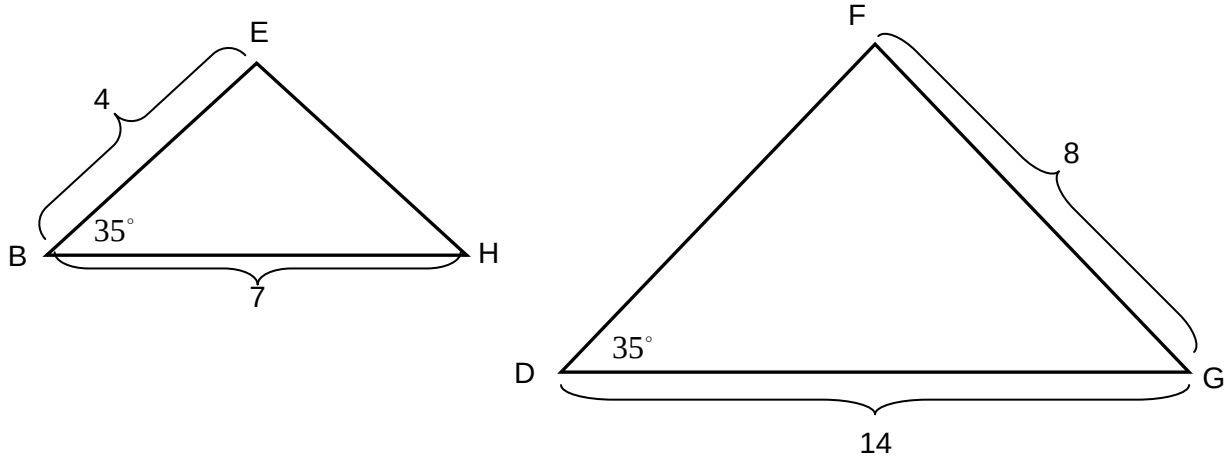
(Can also be written as: $\frac{5}{6} = \frac{10}{12}$ since both reduce to $\frac{5}{6}$)

$$m\angle A = m\angle C = 40°.$$

Therefore $\Delta ABC \sim \Delta XYZ$ by SAS.

Example 3:

Examine the two triangles below and determine if ΔBEH is similar to ΔDFG.



Solution:

We know that $\angle EBH = \angle FDG = 35°$.

The ratio of side BH to side DG is 7:14 or 1:2.

The ratio of side BE to side FG is 4 to 8 or 1:2.

However, $\angle EBH$ is the included angle between sides EB and BH.

There is no information about $\angle FGD$ which is the included angle between sides FG and DG.

Therefore, we cannot use **SAS** to determine that the triangles are similar.

There is no information about the measures of the third sides, EH and FD. Since neither triangle is a special right triangle or an equilateral triangle, we cannot make any assumptions about the measure of the third sides.

Therefore, we have insufficient information to say that the two triangles are similar by **SSS**.

There is insufficient information to prove that $\Delta BEH \sim \Delta DFG$.

The similarity postulate **AA** states that if two angles of one triangle are equal to two angles of another triangle, the two triangles are similar.

If two angles of a triangle are equal to two angles of another triangle, then the third angles of each triangle must also be equal. Can you explain the reason for this?

Example 4:

One triangle has two angles that measure 44° and 55°. A second triangle has two angles that measure 44° and 81°. Are the two triangles similar? Explain your reasoning.

Solution:

The sum of the angles of a triangle total 180°. In the first triangle, the third angle measures $180° - (44° + 55°) = 81°$. Since two angles of the first triangle are the same as two angles of the second triangle, the triangles must be similar by the AA Similarity Postulate.

Example 5:

One angle of a right triangle is 37.5°. One angle of a second right triangle measures 52.5°. Are the two triangles similar?

Solution:

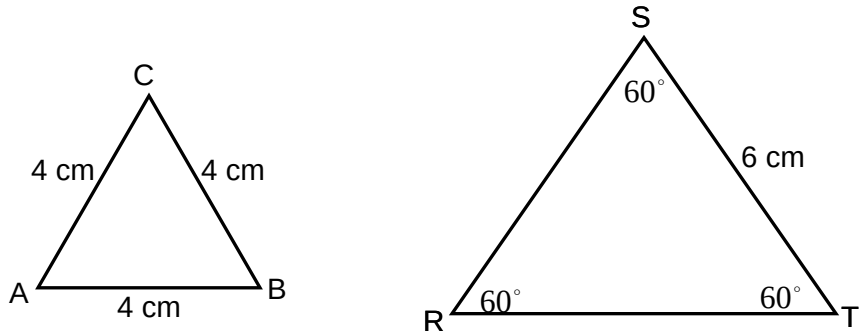
In a right triangle, the two acute angles are complementary. Since 37.5° + 52.5° = 90°, the two triangles have two acute angles that must be the same. All angles of each triangle are the same (37.5°, 52.5°, and 90°).

Since two angles of the first triangle are the same as two angles of the second triangle, the triangles must be similar by the AA Similarity Postulate.

Are right triangles always similar if one acute angle of one right triangle is the same as one acute angle of another right triangle? Justify your answer.

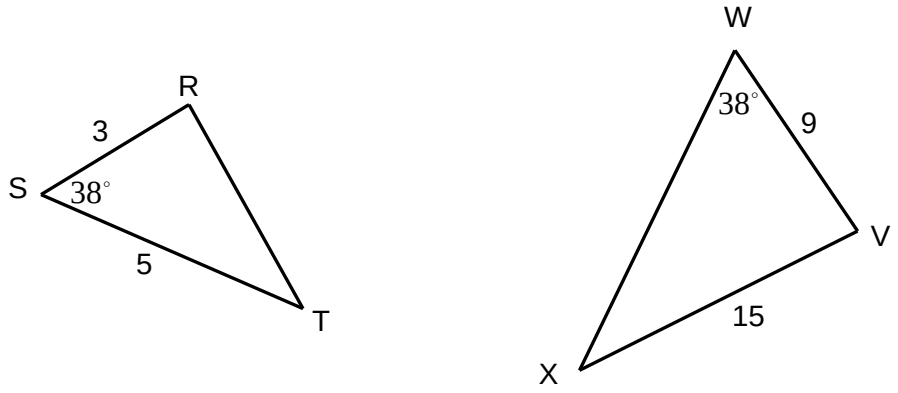
Determine if the following sets of triangles are similar. Justify your answer in the space provided.

1. Each side of ΔABC is 4 cm. In ΔRST, the measure of each angle is 60° and one side is 6 cm. Determine if the two triangles are similar and justify your answer in the space provided.



2. The acute angle of one right triangle is 34°. The acute angle of a second right triangle is 56°. Are the two right triangles similar? Justify your answer.

3. Are the two triangles ΔRST and ΔVWX shown below similar? Justify your answer.



4. Complete the following table showing two triangles with measurements marked on the triangles that make the triangles similar by the given similarity postulate.

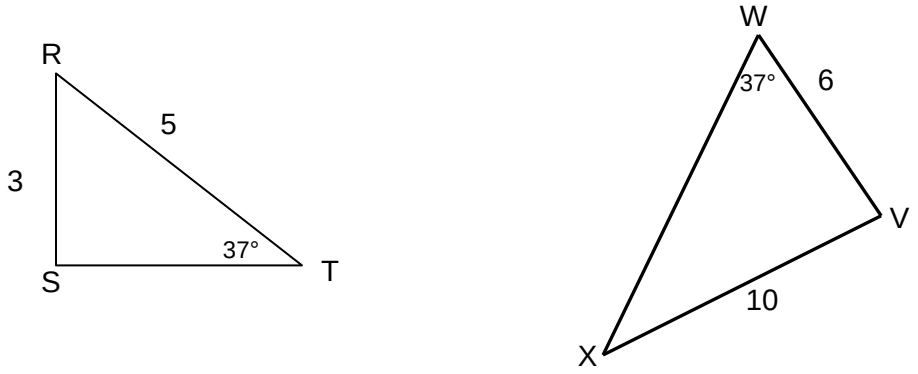
SIMILARITY POSTULATE	EXAMPLE
SSS	
SAS	
AA	

Congruence and Similarity of Triangles
Assessment 2 – Similarity

1. Are two equilateral triangles always similar? Justify your answer.

2. The acute angle of one right triangle is 44°. The acute angle of a second right triangle is 46°. Are the two right triangles similar? Justify your answer.

3. Are the two triangles ΔRST and ΔVWX shown below similar? Justify your answer.



4. Describe the similarity postulates in your own words.

Side-side-side (SSS)

Side-angle-side (SAS)

Angle-angle (AA)

Extensions

1. Examine similar and congruent triangles in architecture.

2. Examine similar and congruent triangles in art.

3. Explore tests used to determine if polygons (other than triangles) are congruent.
   http://www.mathopenref.com/congruentpolygons.html
   http://www.mathopenref.com/congruentpolygonstests.html
   http://www.mathopenref.com/congruenttriangles.html

Sources

1999 Bringing the NCTM Standards to Life, Exemplary Practices from High Schools
2000 NCTM Principles and Standards, p. 308-318
2006 NCTM Curriculum Focal Points, p. 20, 39
2008 AZ Mathematics Standards
2008 The Final Report of the National Mathematics Advisory Panel, p. 17-18, 29

Circle Relationships
An ADE Mathematics Lesson
Days 31-40

Author	ADE Content Specialists
Grade Level	10th grade
Duration	Ten days

Aligns To

Mathematics HS:
Strand 1: Number and Operations
Concept 2: Numerical Operations
PO 3. Calculate powers and roots of rational and irrational numbers.

Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 1. Create and explain the need for equivalent forms of an equation or expression.
PO 2. Solve formulas for specified variables.

Strand 4: Geometry and Measurement
Concept 1: Geometric Shapes
PO 1. Use the basic properties of a circle (relationships between angles, radii, intercepted arcs, chords, tangents, and secants) to prove basic theorems and solve problems.
PO 6. Solve problems using angle and side length relationships and attributes of polygons.
Concept 4: Measurement
PO 2. Find the length of a circular arc; find the area of a sector of a circle.

Strand 5: Structure and Logic
Concept 1: Algorithms and Algorithmic Thinking
PO 1. Select an algorithm that explains a particular mathematical process; determine the purpose of a simple mathematical algorithm.
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.
PO 5. Summarize and communicate mathematical ideas using formal and informal reasoning.
PO 7. Find structural similarities within different algebraic expressions and geometric figures.

Connects To

Mathematics HS:
Strand 1: Number and Operations
Concept 3: Estimation
PO 1. Determine rational approximations of irrational numbers.

Strand 4: Geometry and Measurement
Concept 4: Measurement
PO 1. Use dimensional analysis to keep track of units of measure when converting.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 3. Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.

Aligns To	Connects To
Strand 5: Structure and Logic **Concept 2: Logic, Reasoning, Problem Solving, and Proof** **PO 12.** Construct a simple formal deductive proof. **PO 13.** Identify and explain the roles played by definitions, postulates, propositions and theorems in the logical structure of mathematics, including Euclidean geometry.	

Overview

In this lesson, you will study circles and their relationships to angles, chords, secants, and other geometric properties.

Purpose

Circles form parts of many other relationships and are the basis for many mathematical algorithms such as the unit circle found in trigonometry. It is important to have a good knowledge of the basic properties of a circle and their relationship to chords, tangents, angles, secants, radii, and similar geometric ideas.

Materials

- Circle relationship worksheets
- Ruler/Protractor
- Compass

Objectives

Students will:

- Use the basic properties of a circle (relationships between angles, radii, intercepted arcs, chords, tangents, and secants) to prove basic theorems and solve problems.
- Solve problems using angle and side length relationships and attributes of polygons.
- Find the length of a circular arc; find the area of a sector of a circle.

Lesson Components

Prerequisite Skills: In earlier grades, you learned the fundamentals of circles. You learned how to find the radius, diameter, circumference, and area of a circle. You learned that concentric circles are circles that contain the same center but have different radii. You learned that vertical angles are congruent. You learned how to construct a circle using a compass.

Vocabulary: *circle, central angle, inscribed angle, vertical angle, chord, secant, tangent, radius, intercept arc, sector of a circle*

Session 1 (5 days)

1. Review basic circle relationships and find the measures of central angles, chords, and inscribed angles. Find the area of a sector of a circle.

Session 2 (5 days)

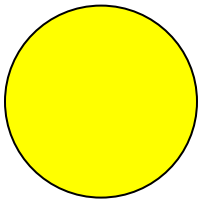
1. Find the measures of an intersecting tangent and its chord, angles formed by two secants or intersecting outside a circle, a tangent and a secant intersecting outside a circle, and two tangents intersecting outside a circle.

Assessment

There are two assessments that will help pinpoint misconceptions before moving on to more complex comparisons. The first assessment comes after Session 1 and addresses basic circle relationships. The second assessment comes after Session 2 and assesses knowledge of lines that intersect either on the circumference of a circle or in a circle.

Circle Relationships
Session 1

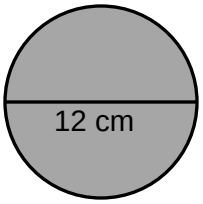
Consider the following circle:



Define each of the following terms in relationship to the circle in your own words.

Term	Definition
Radius	
Diameter	
Circumference	
Center of the circle	
Circumference of circle	
Area of circle	

Example 1: Let's review how to find the circumference and area of a circle. Find the circumference and area of a circle whose diameter is 12 cm.



Solution: If the diameter of the circle is $12\,cm$, the radius of the circle is $\frac{12}{2} = 6\ cm$.

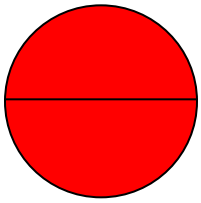
The circumference of the circle is given by either $C = 2\pi r$ or $C = \pi d$.

$$C = \pi d \Rightarrow C = \pi \bullet 12\ cm \Rightarrow C = 12\pi\ \ cm.$$

The area of the circle is given by $A = \pi r^2$.

$$A = \pi \bullet 6^2 = 36\pi\ cm^2$$

Example 2:



A **degree** is a unit of measure based on dividing a circle into 360 equal parts. Degrees are used to measure angles, arcs and rotations.

How many degrees are there in a circle? _____

Each half of a circle is called a **semicircle.** How many degrees are there in a semicircle?

_____

What if we wanted to find the area of the semicircle that had a diameter of 12? Can you think of a way we could find the area?

A **central angle** is an angle whose vertex is the center of a circle and whose sides (rays) are radii.

A **sector of a circle** is a region bounded by a central angle and its arc.

We can find the area of a sector of a circle by setting up a proportion.

$$\frac{\text{central angle}}{360°} = \frac{\text{area}_{\text{sector}}}{\text{area}_{\text{circle}}}$$

To find the area of the semicircle, set up the proportion.

$$\frac{\text{central angle}}{360°} = \frac{\text{area}_{\text{sector}}}{\text{area}_{\text{circle}}} \Rightarrow \frac{180°}{360°} = \frac{\text{area}_{\text{sector}}}{36\pi \text{ cm}^2} \Rightarrow \frac{1}{2} = \frac{\text{area}_{\text{sector}}}{36\pi \text{ cm}^2}$$

$$\mathbf{\frac{1}{2} = \frac{area_{sector}}{36\pi \text{ cm}^2} \Rightarrow 2 \bullet area_{sector} = 36\pi \text{ cm}^2 \Rightarrow area_{sector} = \frac{36\pi \text{ cm}^2}{2}}$$

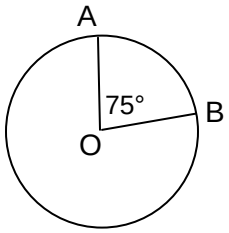
$$\text{area}_{\text{sector}} = 18\pi \text{ cm}^2$$

Summarize how to find the area of a semi-circle.

The measure of a central angle is the measure of its intercepted arc. An **intercepted arc** is that part of a circle that lies between two segments, rays, or secants that intersect the circle.

Example 3:

Find the measure of a central angle $\angle AOB$ whose intercepted arc $\overset{\frown}{AB}$ is 75°.



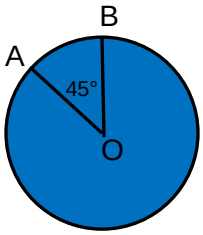
Solution:

The measure of a central angle equals the measure of its intercepted arc.

Therefore, the measure of $\angle AOB$ is 75°.

Example 4:

Find the measure of an intercepted arc, $\overset{\frown}{AB}$ whose central angle $\angle AOB$ is 45°.



Solution:

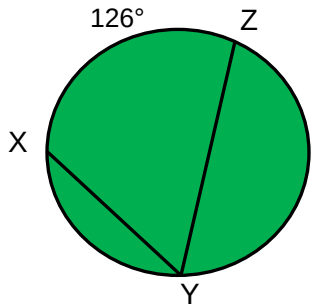
The central angle equals the measure of its intercepted arc. Therefore, the measure of $\overset{\frown}{AB}$ is 45°.

Example 5:

A **chord** is a segment whose endpoints are on a given circle. An **inscribed angle** is an angle with its vertex on the circle and with sides (rays) that are chords of the circle.

The measure of an inscribed angle is one half of its intercepted arc.

Find the measure of inscribed angle, $\angle XYZ$.



Solution:

The measure of the inscribed angle $\angle XYZ$ $(m\angle XYZ)$ equals one half the measure of its intercepted arc, $\overset{\frown}{XZ}$.

$\frac{1}{2} \bullet 126° = 63°$. Therefore, $m\angle XYZ = 63°$.

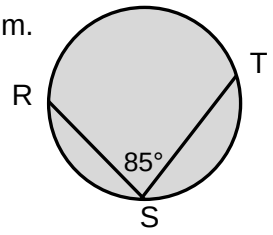
Example 6:

The measure of an inscribed angle of a circle is $85°$.

- What is the measure of its intercepted arc?
- What is the measure of any central angle with the same intercepted arc?

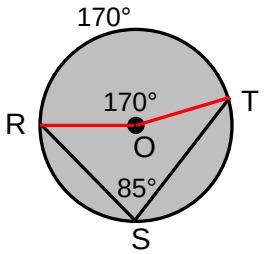
Solution:

First draw a picture to illustrate the problem.



Since the measure of the inscribed angle is one half of its intercepted arc, the measure of an intercepted arc would be twice its inscribed angle.

The measure of $\overset{\frown}{RT}$ is 85° • 2 = 170°.



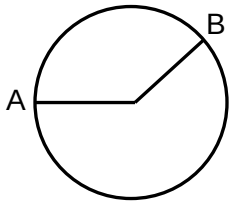
The measure of a central angle is equal to the measure of its intercepted arc. Therefore, the measure of any central angle with an intercepted arc of $170°$ would be $170°$.

$$m\angle ROT = 170°$$

Now it's your turn to try some problems. Show your work in the space provided and draw diagrams when necessary to help you solve a problem.

1. Find the area and circumference of a circle whose diameter is 8 cm.

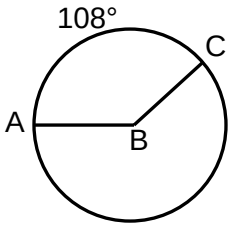
2. How many degrees are there in the angle shown in the diagram if the intercepted arc $\overset{\frown}{AB}$ is $\frac{1}{3}$ of the circumference of the circle?



3. Find the measure of a central angle whose intercepted arc is 52°.

4. Find the measure of an intercepted arc, $\overset{\frown}{AB}$ whose central angle $\angle AOB$ is 71°.

5. Find the measure of inscribed angle $\angle ABC$.



6. The measure of an inscribed angle of a circle is 46°.

- What is the measure of its intercepted arc?
- What is the measure of any central angle with the same intercepted arc?

Circle Relationships
Assessment 1

Problem 1: Define the following terms in your own words.

Term	Definition
Chord	
Central angle	
Inscribed angle	
Intercepted arc	
Degree	
Semicircle	
Sector of a circle	

Problem 2:

Find the area of a sector of a circle whose central angle measures $45°$ and whose radius is 7 cm. Draw a diagram and show your work in the space provided.

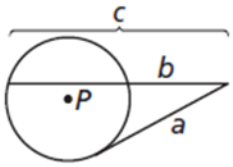
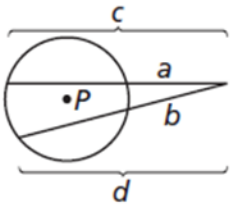
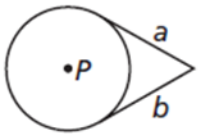
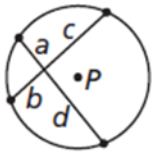
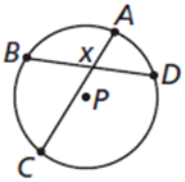
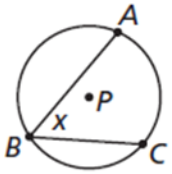
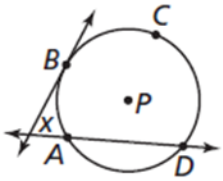
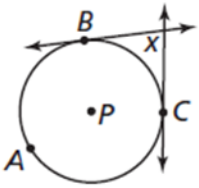
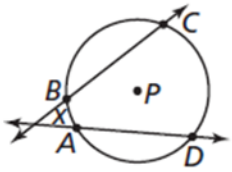
Problem 3:

An inscribed angle has a measure of $23°$. Find both its intercepted arc and any central angle with that intercepted arc. Draw a diagram and show your work in the space provided.

Circle Relationships
Session 2

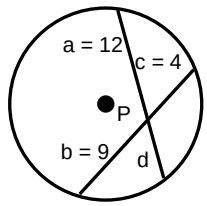
Circle Definitions	
Major Arc of a circle	an arc whose measure is greater than 180 degrees
Minor Arc of a circle	an arc whose measure is less than 180 degrees
Chord of a circle	a segment whose endpoints are on a given circle
Secant of a circle	a line that intersects a circle at two points
Sector of a circle	a region bounded by a central angle and its arc
Tangent of a circle	a line in the plane of a circle that intersects the circle at exactly one point

Circle Formulas

 $\frac{c}{a} = \frac{a}{b}$ or $a^2 = bc$	 $\frac{d}{a} = \frac{c}{b}$ or $ac = bd$	 $a = b$
 $\frac{a}{b} = \frac{c}{d}$ or $ad = bc$	 $m\angle x = \frac{1}{2}\left(m\widehat{AB} + m\widehat{CD}\right)$	 $m\angle x = \frac{1}{2}m\widehat{AC}$
 $m\angle x = \frac{1}{2}\left(m\widehat{BCD} - m\widehat{AB}\right)$	 $m\angle x = \frac{1}{2}\left(m\widehat{BAC} - m\widehat{BC}\right)$	 $m\angle x = \frac{1}{2}\left(m\widehat{CD} - m\widehat{AB}\right)$

In many instances, we need to find the length of a chord, tangent, or secant. There are several different formulas to do this. Some of the circle formulas are just special cases of a more general form.

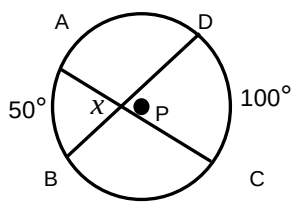
Example 1: Let's examine two chords that intersect in the interior of the circle. Find the measure of the missing chord in the diagram.



Solution: We know that we can set up a proportion according to the formula $\frac{a}{b} = \frac{c}{d}$ or that $ad = bc$. Therefore, it follows that $ad = bc \Rightarrow 12 \bullet d = 9 \bullet 4 \Rightarrow 12d = 36 \Rightarrow d = 3$.

We could have also set up the proportion $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{12}{9} = \frac{4}{d} \Rightarrow 12d = 36 \Rightarrow d = 3$.

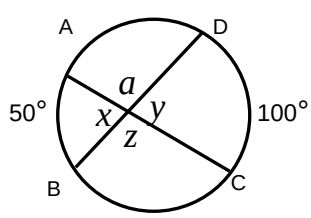
Example 2: Find the measure of $\angle x$ in circle P below.



Solution: $m\angle x = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$

$$m\angle x = \frac{1}{2}(50° + 100°) = \frac{1}{2}(150°) = 75°$$

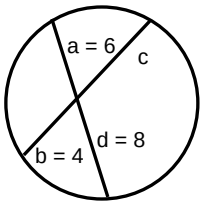
Note that we now know the measures of all the angles shown in this figure.



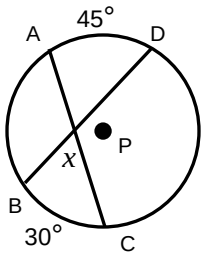
Since vertical angles are congruent, it follows that $m\angle x = m\angle y = 75°$ and $m\angle z = m\angle a = 105°$.

Now you try some problems:

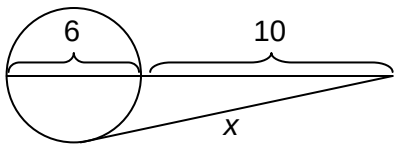
Problem 1: Find the measure of the missing chord in the diagram. Show your work in the space provided.



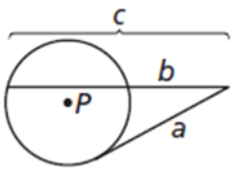
Problem 2: Find the measure of $\angle x$ in circle P below.



Example 3: Find x in the picture below.



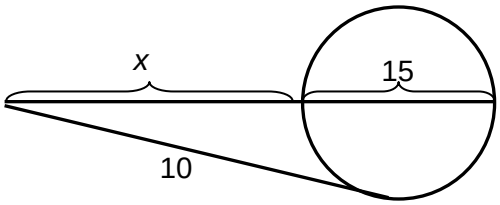
Solution: The diagram and formula below shows the relationship between the lengths of the line segments.



$\frac{c}{a} = \frac{a}{b}$ or $a^2 = bc$

$$\frac{10+6}{x} = \frac{x}{10} \Rightarrow (10+6)\bullet 10 = x \bullet x \Rightarrow 16 \bullet 10 = x^2 \Rightarrow x^2 = 160 \Rightarrow x = 4\sqrt{10}$$

Example 4:



Solution:

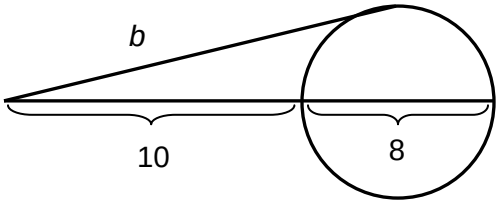
$$\frac{x+15}{10} = \frac{10}{x} \Rightarrow (x+15)\bullet x = 10 \bullet 10 \Rightarrow x^2 + 15x = 100 \Rightarrow x^2 + 15x - 100 = 0$$

$$x^2 + 15x - 100 = 0 \Rightarrow (x+20)(x-5) = 0$$

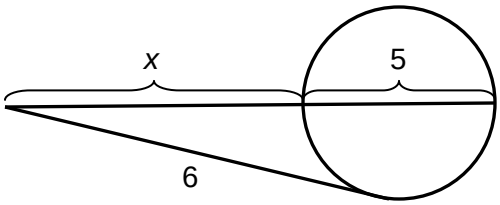
$$(x+20)(x-5) = 0 \Rightarrow x + 20 = 0 \text{ and } x - 5 = 0 \Rightarrow x = -20,\ 5$$

Since *x* is a distance it cannot be negative so the correct answer is $x = 5$.

Problem 3: Find the value of b in the diagram below. Show your work in the space provided.

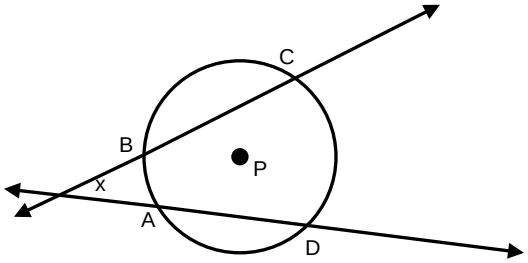


Problem 4: Find the measure of x in the diagram below. Show all your work in the space provided.



Example 5:

Find the measure of $\angle x$ in the diagram below. $m\overset{\frown}{CD}$ = 100° and $m\overset{\frown}{AB}$ = 25°

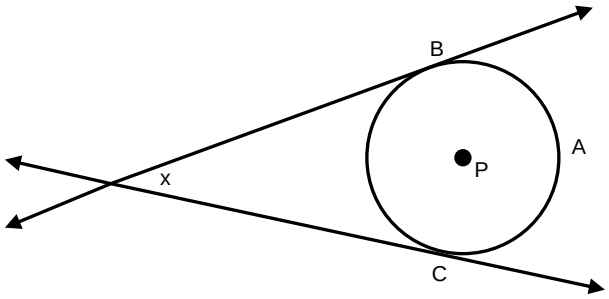


Solution:

$$m\angle x = \frac{1}{2}(m\overset{\frown}{CD} - m\overset{\frown}{AB}) \Rightarrow\ m\angle x = \frac{1}{2}(100° - 25°) \Rightarrow\ m\angle x = \frac{1}{2}(75°) = 37.5°$$

Example 6:

Find the measure of $\angle x$ in the diagram below. $m\overset{\frown}{BAC}$ = 236° and $m\overset{\frown}{BC}$ = 124°

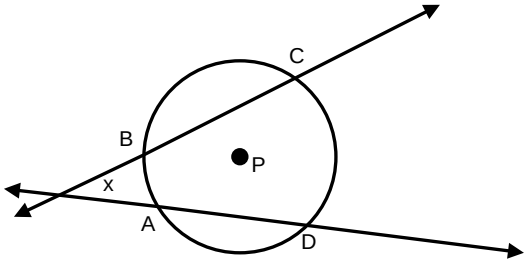


Solution:

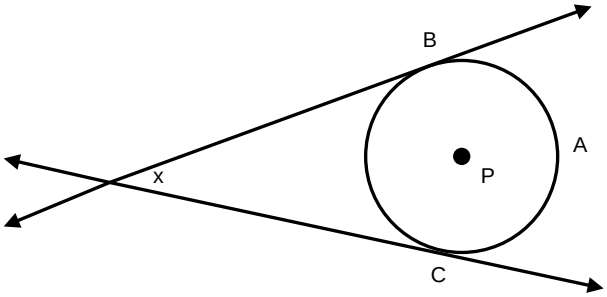
$$m\angle x = \frac{1}{2}m\overset{\frown}{BAC} - m\overset{\frown}{BC} \Rightarrow\ m\angle x = \frac{1}{2}(236° - 124°)\ \Rightarrow\ m\angle x = \frac{1}{2}(112°) = 56°$$

Problem 5: Find the measure of $\angle x$ in the diagram below. $m\overset{\frown}{CD} = 125°$ and $m\overset{\frown}{AB} = 37°$

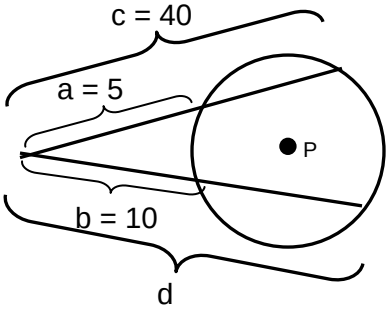
Show your work in the space provided.



Problem 6: Find the measure of $\angle x$ in the diagram below. $m\overset{\frown}{BAC} = 222°$ and $m\overset{\frown}{BC} = 138°$



Example 7: Find the measure of *d* in the diagram below if a = 5, b = 10, and c = 40.



Solution: $\frac{d}{a} = \frac{c}{b} \Rightarrow ac = bd \Rightarrow 5 \bullet 40 = 10 \bullet d \Rightarrow 200 = 10d \Rightarrow d = 20$

We have used formulas that involve chords, secants and tangents. Define each of these terms in your own words in the space provided.

Term	**Definition**
Chord	
Tangent	
Secant	

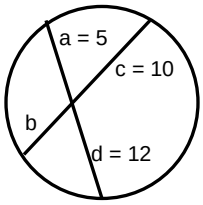
Circle Relationships
Assessment 2

1. Complete the following chart justifying your answers in the space provided.

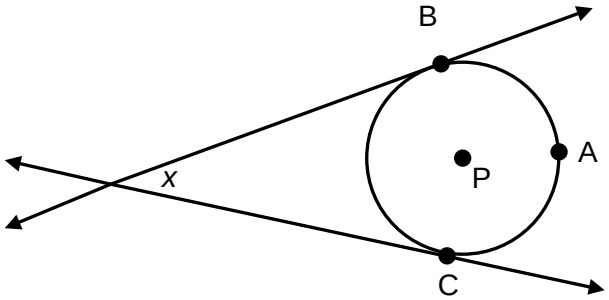
The difference between a chord and a secant is
The difference between a central angle and an inscribed angle is
The difference between a tangent and a secant is
The difference between a major arc and a minor arc is

Find the measures of the missing chords, tangents, or secants in the following problems showing all your work in the space provided.

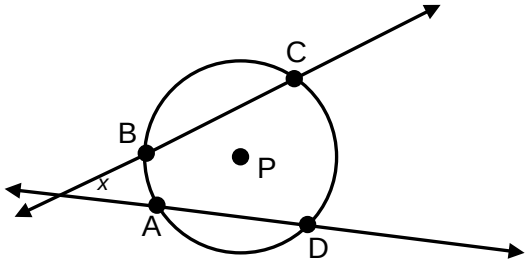
1. Find the measure of the missing chord in the diagram.



2. Find the measure of $\angle x$ in the diagram below. $\overset{\frown}{BAC} = 205°$ and $\overset{\frown}{BC} = 155°$.



3. Find the measure of $\angle x$ in the diagram below. $\overset{\frown}{CD} = 126°$ and $\overset{\frown}{AB} = 33°$.



Extensions

1. Find central angles, inscribed angles, intercepted arcs, etc. when the measurements are given as algebraic expressions.

2. Explore relationships between central angles and intercepted arcs using an interactive applet.

   http://www.analyzemath.com/Geometry/CentralInscribedAngle/CentralInscribedAngle.html

Sources

1999 Bringing the NCTM Standards to Life, Exemplary Practices from High Schools, p. 9-16
2000 NCTM Principles and Standards, p. 308-318
2008 AZ Department of Education Mathematics Standards
2008 The Final Report of the National Mathematics Advisory Panel, p. 29-30